Turbulence in the Presence of a Vertical Body Force and Temperature Gradient[1]

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Abstract. Two-point correlation equations which include the effects of a uniform temperature gradient and body force are constructed from the Navier-Stokes, heat-transfer, and continuity equations. A solution is obtained by converting the correlation equations to spectral form and assuming that the turbulence is sufficiently weak for triple correlations to be neglected. It is shown that the turbulence decays with time, although the rate of decay is altered by buoyancy effects caused by the body force and temperature gradient. The buoyancy forces can either extract energy from the turbulent field or feed energy into it, depending on the directions of the body force and temperature gradient. Spectra are calculated for the turbulent energy and for the various terms in the turbulent energy equation as well as for the temperature fluctuations and turbulent heat transfer. For fluids with Prandtl numbers less than 1 the buoyancy forces act mainly on the large eddies, whereas for higher Prandtl numbers they can act on the smaller ones. When the buoyancy forces are stabilizing, the turbulence can cause heat to flow against the temperature gradient for certain values of the parameters. For making the calculations, it is assumed that the turbulence is initially isotropic and the temperature fluctuations initially zero.

INTRODUCTION

The work described here is concerned with the effect of buoyancy forces on a homogeneous turbulent field. The buoyancy effects are produced by a uniform vertical temperature gradient and body force. To make the problem tractable, we assume that the turbulence is weak enough for triple correlations to be negligible in comparison with double correlations. Other studies of low-Reynolds-number turbulence are given in *von Kármán and Howarth* [1938], *Batchelor and Townsend* [1948], *Dunn and Reid* [1958], *Pearson* [1959], and *Deissler* [1961*a* and *b*]. An experimental basis for low-Reynolds-number solutions is given by *Batchelor and Townsend* [1948]. Although the transfer of energy between eddies of various sizes is not present when triple correlations are neglected (except in the presence of a mean velocity gradient [*Deissler*, 1961*a*]), it appears that the analysis can provide insight about the other important turbulent processes, such as the dissipation and the production or extraction of turbulent energy by buoyancy forces.

To proceed with the analysis we require two-point correlation equations which include buoyancy effects. These will be derived in the next section.

CORRELATION AND SPECTRAL EQUATIONS

The Navier-Stokes equation with a body force is

$$\frac{\partial u_i}{\partial t} + \frac{\partial(u_i u_k)}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_k\,\partial x_k} + g_i \quad (1)$$

where the subscripts can take on the values 1, 2, or 3 and a repeated subscript in a term indicates a summation. The quantity u_i is an instantaneous velocity component, x_i a space coordinate, t the time, ρ the density, ν the kinematic viscosity, p the instantaneous pressure, and g_i a component of the body force. If the density depends, effectively, only on temperature and is not far removed from its equilibrium value (value it would have for no heat transfer or turbulence), equation 1 can be written as [*Landau and Lifshitz*, 1959],

$$\frac{\partial u_i}{\partial t} + \frac{\partial(u_i u_k)}{\partial x_k} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}(p - p_e) + \nu\frac{\partial^2 u_i}{\partial x_k\,\partial x_k} - \beta(\tilde{T} - T_e)g_i \quad (2)$$

where $\tilde{T}$ is the instantaneous temperature, T_e and p_e are, respectively, the equilibrium temperature and pressure, and β is the thermal expansion coefficient given by

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$$\beta \equiv -(1/\rho)(\partial\rho/\partial T)_p \tag{3}$$

Equation 2 applies at a point in the fluid, say point P. A similar equation written for a point P', separated from the point P by the vector $\mathbf{r}$, is

$$\frac{\partial u_i'}{\partial t} + \frac{\partial(u_i'u_k')}{\partial x_k'} = -\frac{1}{\rho}\frac{\partial}{\partial x_i'}(p' - p_e') + \nu\frac{\partial^2 u_i'}{\partial x_k'\,\partial x_k'} - \beta(\tilde{T}' - T_e)g_i \tag{4}$$

Note that the equilibrium temperature is uniform, whereas the equilibrium pressure is not. The instantaneous temperature can be written as

$$\tilde{T} = T + \tau \tag{5}$$

where T is the time or ensemble average temperature and τ is the fluctuating component of the temperature. Multiplying equation 2 by u'_i and equation 4 by u_i, adding, taking averages, and substituting equation 5 give

$$\frac{\partial\overline{u_iu_i'}}{\partial t} + \frac{\partial\overline{u_iu_ku_i'}}{\partial x_k} + \frac{\partial\overline{u_iu_i'u_k'}}{\partial x_k'} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{pu_i'} - \frac{1}{\rho}\frac{\partial}{\partial x_i'}\overline{u_ip'} + \nu\frac{\partial^2\overline{u_iu_i'}}{\partial x_k\,\partial x_k} + \nu\frac{\partial^2\overline{u_iu_i'}}{\partial x_k'\,\partial x_k'} - \beta g_i\overline{\tau u_i'} - \beta g_i\overline{u_i\tau'} \tag{6}$$

where the overbars indicate average values. In obtaining equation 6, the fact that fluctuating quantities at one point are independent of the position of the other point was used. Introducing the variable $r_i \equiv x'_i - x_i$ gives, for homogeneous turbulence,

$$\frac{\partial}{\partial t}\overline{u_iu_i'} + \frac{\partial}{\partial r_k}(\overline{u_iu_i'u_k'} - \overline{u_iu_ku_i'}) = -\frac{1}{\rho}\left(\frac{\partial}{\partial r_i}\overline{u_ip'} - \frac{\partial}{\partial r_i}\overline{pu_i'}\right) + 2\nu\frac{\partial^2\overline{u_iu_i'}}{\partial r_k\,\partial r_k} - \beta g_i\overline{\tau u_i'} - \beta g_i\overline{u_i\tau'} \tag{7}$$

Equation 7, except for the last two terms, which give the effect of buoyancy forces, is the same as that obtained by *von Kármán and Howarth* [1938].

An expression for the pressure-velocity correlations can be obtained by taking the divergence of equation 2 and using the continuity relation, which is, for small density variations, $\partial u_i/\partial x_i = 0$. This gives

$$\frac{1}{\rho}\frac{\partial^2(p - p_e)}{\partial x_i\,\partial x_i} = -\beta g_i\left(\frac{\partial T}{\partial x_i} + \frac{\partial\tau}{\partial x_i}\right) - \frac{\partial^2(u_iu_k)}{\partial x_k\,\partial x_i} \tag{8}$$

Multiplying equation 8 by u'_j, taking averages, and introducing the variable r_i result in

$$\frac{1}{\rho}\frac{\partial^2\overline{pu_j'}}{\partial r_i\,\partial r_i} = \beta g_i\frac{\partial\overline{\tau u_j'}}{\partial r_i} - \frac{\partial^2\overline{u_iu_ku_j'}}{\partial r_k\,\partial r_i} \tag{9}$$

Similarly, from equation 4,

$$\frac{1}{\rho}\frac{\partial^2\overline{u_ip'}}{\partial r_j\,\partial r_j} = -\beta g_j\frac{\partial\overline{u_i\tau'}}{\partial r_j} - \frac{\partial^2\overline{u_iu_j'u_k'}}{\partial r_k\,\partial r_j} \tag{10}$$

To obtain expressions for the temperature-velocity correlations in equations 7, 9, and 10, the heat-transfer equation must be considered. When frictional heating is neglected, this equation can be written for points P and P' as

$$\frac{\partial\tilde{T}}{\partial t} + \frac{\partial(\tilde{T}u_k)}{\partial x_k} = \frac{\partial^2\tilde{T}}{\partial x_k\,\partial x_k} \tag{11}$$

and

$$\frac{\partial\tilde{T}'}{\partial t} + \frac{\partial(\tilde{T}'u_k')}{\partial x_k'} = \alpha\frac{\partial^2\tilde{T}'}{\partial x_k'\,\partial x_k'} \tag{12}$$

Substituting equation 5 in equations 11 and 12, averaging, and subtracting the averaged equations from the unaveraged ones give

$$\frac{\partial\tau}{\partial t} + u_k\frac{\partial T}{\partial x_k} + \frac{\partial(u_k\tau)}{\partial x_k} - \frac{\partial\overline{u_k\tau}}{\partial x_k} = \alpha\frac{\partial^2\tau}{\partial x_k\,\partial x_k} \tag{13}$$

$$\frac{\partial\tau'}{\partial t} + u_k'\frac{\partial T'}{\partial x_k'} + \frac{\partial(u_k'\tau')}{\partial x_k'} - \frac{\partial\overline{u_k'\tau'}}{\partial x_k'} = \alpha\frac{\partial^2\tau'}{\partial x_k'\,\partial x_k'} \tag{14}$$

Multiplying equation 13 by u', and equation 4 by τ, adding, taking averages, substituting equation 5, and introducing the variable r_i give, for a uniform temperature gradient,

$$\frac{\partial\overline{\tau u_j'}}{\partial t} + \overline{u_ku_j'}\frac{\partial T}{\partial x_k} + \frac{\partial}{\partial r_k}(\overline{\tau u_j'u_k'} - \overline{\tau u_ku_j'}) = -\frac{1}{\rho}\frac{\partial\overline{\tau p'}}{\partial r_j} + (\alpha + \nu)\frac{\partial^2\overline{\tau u_j'}}{\partial r_k\,\partial r_k} - \beta g_j\overline{\tau\tau'} \tag{15}$$

Similarly, from equations 14, 2, and 5,

$$\frac{\partial \overline{v_i \tau'}}{\partial t} + \frac{\partial}{\partial r_k}(\overline{u_i u_k' \tau'} - \overline{u_i u_k \tau'}) + \overline{u_i u_k'} \frac{\partial T}{\partial x_k} = \frac{1}{\rho} \frac{\partial}{\partial r_i} \overline{p\tau'} + (\alpha + \nu) \frac{\partial^2 \overline{u_i \tau'}}{\partial r_k \, \partial r_k} - \beta g_i \overline{\tau\tau'} \quad (16)$$

An expression for $\overline{\tau\tau'}$ can be obtained from equations 13 and 14:

$$\frac{\partial \overline{\tau\tau'}}{\partial t} + (\overline{u_k \tau'} + \overline{\tau u_k'}) \frac{\partial T}{\partial x_k} + \frac{\partial}{\partial r_k}(\overline{\tau\tau' u_k'} - \overline{u_k \tau\tau'}) = 2\alpha \frac{\partial^2 \overline{\tau\tau'}}{\partial r_k \, \partial r_k} \quad (17)$$

Expressions for the pressure-temperature correlations in equation 16 are obtained from equation 8 and a similar equation for p':

$$\frac{1}{\rho} \frac{\partial^2 \overline{p\tau'}}{\partial r_i \, \partial r_i} = \beta g_i \frac{\partial \overline{\tau\tau'}}{\partial r_i} - \frac{\partial^2 \overline{u_i u_k \tau'}}{\partial r_i \, \partial r_k} \quad (18)$$

$$\frac{1}{\rho} \frac{\partial^2 \overline{\tau p'}}{\partial r_j \, \partial r_j} = -\beta g_j \frac{\partial \overline{\tau\tau'}}{\partial r_j} - \frac{\partial^2 \overline{\tau u_j' u_k'}}{\partial r_k \, \partial r_j} \quad (19)$$

If the turbulence is sufficiently weak for triple correlations to be neglected, equations 7, 9, 10, 15, 16, 17, 18, and 19 form a determinate set. It is desirable to write the equations in spectral form in order to reduce them to ordinary differential or algebraic equations and because of the physical significance of spectral quantities. For this purpose we can introduce three-dimensional Fourier transforms defined as follows:

$$\overline{u_i u_j'} = \int_{-\infty}^{\infty} \varphi_{ij} \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) \, d\boldsymbol{\kappa} \quad (20)$$

$$\overline{p u_j'} = \int_{-\infty}^{\infty} \lambda_j \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) \, d\boldsymbol{\kappa} \quad (21)$$

$$\overline{u_i p'} = \int_{-\infty}^{\infty} \lambda_i' \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) \, d\boldsymbol{\kappa} \quad (22)$$

$$\overline{p\tau'} = \int_{-\infty}^{\infty} \zeta \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) \, d\boldsymbol{\kappa} \quad (23)$$

$$\overline{\tau p'} = \int_{-\infty}^{\infty} \zeta' \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) \, d\boldsymbol{\kappa} \quad (24)$$

$$\overline{\tau u_j'} = \int_{-\infty}^{\infty} \gamma_j \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) \, d\boldsymbol{\kappa} \quad (25)$$

$$\overline{u_i \tau'} = \int_{-\infty}^{\infty} \gamma_i' \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) \, d\boldsymbol{\kappa} \quad (26)$$

$$\overline{\tau\tau'} = \int_{-\infty}^{\infty} \delta \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) \, d\boldsymbol{\kappa} \quad (27)$$

where $\boldsymbol{\kappa}$ is a wave-number vector having the dimension 1/length and $d\boldsymbol{\kappa} = d\kappa_1 \, d\kappa_2 \, d\kappa_3$. Substituting these Fourier transforms into equations 7, 9, 10, 15, 16, 17, 18, and 19 and neglecting triple correlations result in

$$\frac{\partial \varphi_{ij}}{\partial t} = -\frac{1}{\rho}(i\kappa_j \lambda_i' - i\kappa_i \lambda_j) - 2\nu\kappa^2 \varphi_{ij} - \beta g_i \gamma_j - \beta g_j \gamma_i' \quad (28)$$

$$-\frac{1}{\rho} \kappa^2 \lambda_j = \beta g_k i\kappa_k \gamma_j \quad (29)$$

$$-\frac{1}{\rho} \kappa^2 \lambda_i' = -\beta g_k i\kappa_k \gamma_i' \quad (30)$$

$$\frac{\partial \gamma_j}{\partial t} = -\varphi_{kj} \frac{\partial T}{\partial x_k} - \frac{1}{\rho} i\kappa_j \zeta' - (\alpha + \nu)\kappa^2 \gamma_j - \beta g_j \delta \quad (31)$$

$$\frac{\partial \gamma_i'}{\partial t} = -\varphi_{ik} \frac{\partial T}{\partial x_k} + \frac{1}{\rho} \kappa_i \zeta - (\alpha + \nu)\kappa^2 \gamma_i' - \beta g_i \delta \quad (32)$$

$$-\frac{1}{\rho} \kappa^2 \zeta = \beta g_k i\kappa_k \delta \quad (33)$$

$$-\frac{1}{\rho} \kappa^2 \zeta' = -\beta g_k i\kappa_k \delta \quad (34)$$

$$\frac{\partial \delta}{\partial t} = -(\gamma_k' + \gamma_k) \frac{\partial T}{\partial x_k} - 2\alpha\kappa^2 \delta \quad (35)$$

Substitution of equations 29 and 30 into 28 and equations 33 and 34 into 32 and 31 shows that $\varphi_{ij} = \varphi_{ji}$ and $\gamma_i = \gamma'_i$ for all times if they are equal at an initial time. Here it will be assumed that the turbulence is initially isotropic and that the temperature fluctuations are initially zero, so that the above relations will hold. Thus the set of equations 28 to 35 becomes

$$\frac{\partial \varphi_{ij}}{\partial t} = \beta g_k \frac{\kappa_k \kappa_j}{\kappa^2} \gamma_i + \beta g_k \frac{\kappa_k \kappa_i}{\kappa^2} \gamma_j - 2\nu\kappa^2 \varphi_{ij} - \beta g_i \gamma_j - \beta g_j \gamma_i \quad (36)$$

$$\frac{\partial \gamma_i}{\partial t} = -\varphi_{ki} \frac{\partial T}{\partial x_k} + \beta g_k \frac{\kappa_k \kappa_i}{\kappa^2} \delta - (\alpha + \nu)\kappa^2 \gamma_i - \beta g_i \delta \quad (37)$$

$$\frac{\partial \delta}{\partial t} = -2\gamma_k \frac{\partial T}{\partial x_k} - 2\alpha\kappa^2 \delta \tag{38}$$

Assume that the only nonzero component of **g** is in the negative vertical direction, and let

$$g \equiv -g_3 \tag{39}$$

Also, assume that the uniform temperature gradient is in the vertical direction, and let

$$b \equiv \partial T/\partial x_3 \tag{40}$$

Letting $i = j = 3$ in equations 36, 37, and 38,

$$\frac{d\varphi_{33}}{dt} = -2\beta g \frac{\kappa_3^2}{\kappa^2}\gamma_3 - 2\nu\kappa^2\varphi_{33} + 2\beta g\gamma_3 \tag{41}$$

$$\frac{d\gamma_3}{dt} = -b\varphi_{33} - \beta g \frac{\kappa_3^2}{\kappa^2}\delta - (\alpha+\nu)\kappa^2\gamma_3 + \beta g\,\delta \tag{42}$$

$$\frac{d\delta}{dt} = -2b\gamma_3 - 2\alpha\kappa^2\delta \tag{43}$$

Contracting i and j in equation 36 gives

$$\frac{d\varphi_{ii}}{dt} = -2\nu\kappa^2\varphi_{ii} + 2\beta g\gamma_3 \tag{44}$$

The pressure term (second term in equation 41) drops out of equation 44, as can be seen from equation 7 and the relations $\partial/\partial r_i = -\partial/\partial x_i$ and $\partial/\partial r_i = \partial/\partial x'_i$. Thus, as in the case of homogeneous turbulence without buoyancy effects, the pressure term transfers energy between the directional components of the energy but gives no contribution to the change of energy at a particular wave number.

Solution of Spectral Equations

A general solution of the simultaneous equations 41, 42, and 43 is

$$\varphi_{33} = C_1 \exp[-(\alpha+\nu)\kappa^2(t-t_0)] + C_2 \exp\{-[(\alpha+\nu)\kappa^2 - s](t-t_0)\} + C_3 \exp\{-[(\alpha+\nu)\kappa^2 + s](t-t_0)\} \tag{45}$$

$$\gamma_3 = -\Big(C_1(\alpha-\nu)\kappa^2 \exp[-(\alpha+\nu)\kappa^2(t-t_0)] + C_2[(\alpha-\nu)\kappa^2 - s]\cdot\exp\{-[(\alpha+\nu)\kappa^2 - s](t-t_0)\} + C_3[(\alpha-\nu)\kappa^2 + s]\cdot\exp\{-[(\alpha+\nu)\kappa^2 + s](t-t_0)\}\Big) \div \left[2\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\right] \tag{46}$$

$$\delta = \Bigg(2C_1 b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\cdot\exp[-(\alpha+\nu)\kappa^2(t-t_0)] + C_2\left[(\alpha-\nu)^2\kappa^4 - (\alpha-\nu)\kappa^2 s - 2b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\right]\cdot\exp\{-[(\alpha+\nu)\kappa^2 - s](t-t_0)\} + C_3\left[(\alpha-\nu)^2\kappa^4 + (\alpha-\nu)\kappa^2 s - 2b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\right]\cdot\exp\{-[(\alpha+\nu)\kappa^2 + s](t-t_0)\}\Bigg) \div \left[2\beta^2 g^2\left(1-\frac{\kappa_3^2}{\kappa^2}\right)^2\right] \tag{47}$$

where

$$s \equiv \sqrt{(\alpha-\nu)^2\kappa^4 - 4b\beta g(1-\kappa_3^2/\kappa^2)} \tag{48}$$

and C_1, C_2, and C_3 are constants of integration. Equation 45 indicates that, as t approaches infinity, φ_{33} approaches zero, so that no nonzero steady-state solution exists.

For determining the constants of integration, we use the initial conditions that, for $t = t_0$, the turbulence is isotropic, and $\gamma_3 = \delta = 0$. The last two conditions correspond to the assumption that the temperature fluctuations are zero at $t = t_0$. This would be true, for instance, if the turbulence were produced by an unheated grid. The mean temperature gradient would then cause temperature fluctuations to arise at subsequent times. The assumption that the turbulence is isotropic at $t = t_0$ implies that, for weak turbulence,

$$(\varphi_{ij})_0 = (J_0/12\pi^2)(\kappa^2\delta_{ij} - \kappa_i\kappa_j) \tag{49}$$

as given by equation 43 in *Deissler* [1961*a*]. The turbulence is not, of course, isotropic at subsequent times, as will be seen. By using these initial conditions, the constants of integration are found to be

$$C_1 = -\frac{J_0\kappa^2 b\beta g(1-\kappa_3^2/\kappa^2)^2}{6\pi^2 s^2} \tag{50}$$

$$C_2 = \left(J_0\kappa^2\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\left[(\alpha-\nu)^3\kappa^6 + (\alpha-\nu)^2\kappa^4 s - 4(\alpha-\nu)\kappa^2 b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right) - 2b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)s\right]\right) \div (24\pi^2 s^3) \qquad (51)$$

$$C_3 = -\left(J_0\kappa^2\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\left[(\alpha-\nu)^3\kappa^6 - (\alpha-\nu)^2\kappa^4 s - 4(\alpha-\nu)\kappa^2 b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right) + 2b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)s\right]\right) \div (24\pi^2 s^3) \qquad (52)$$

For small values of κ, the quantity s, as given by equation 48, becomes imaginary. In that case the following solution can be used:

$$\varphi_{33} = \exp\left[-(\alpha+\nu)\kappa^2(t-t_0)\right] \cdot \{C_1' + C_2' \cos[s'(t-t_0)] + C_3' \sin[s'(t-t_0)]\} \qquad (53)$$

$$\gamma_3 = -\left(\exp\left[-(\alpha+\nu)\kappa^2(t-t_0)\right] \cdot \{C_1'(\alpha-\nu)\kappa^2 + [C_2'(\alpha-\nu)\kappa^2 - C_3's'] \cos[s'(t-t_0)] + [C_3'(\alpha-\nu)\kappa^2 + C_2 s'] \sin[s'(t-t_0)]\}\right) \div 2\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right) \qquad (54)$$

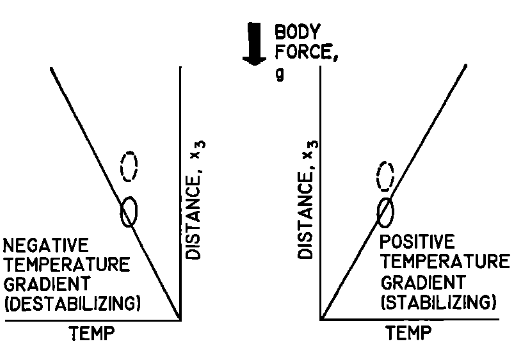

Fig. 1. **Expected** effects of buoyancy forces on turbulent eddy.

$$\delta = \Bigg(\exp -[(\alpha+\nu)\kappa^2(t-t_0)] \cdot \Bigg\{2C_1' b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right) + \left[C_2'(\alpha-\nu)^2\kappa^4 - C_3's'(\alpha-\nu)\kappa^2 - 2C_2' b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\right]\cos[s'(t-t_0)] + \left[C_3'(\alpha-\nu)^2\kappa^4 + C_2's'(\alpha-\nu)\kappa^2 - 2C_3' b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\right]\sin[s'(t-t_0)]\Bigg\}\Bigg) \div \left[2\beta^2 g^2\left(1-\frac{\kappa_3^2}{\kappa^2}\right)^2\right] \qquad (55)$$

where

$$s' \equiv \sqrt{4b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right) - (\alpha-\nu)^2\kappa^4} \qquad (56)$$

$$C_1' = \left[J_0\kappa^2 b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right)^2\right] \div (6\pi^2 s'^2) \qquad (57)$$

$$C_2' = J_0\kappa^2\left(1-\frac{\kappa_3^2}{\kappa^2}\right)\left[2b\beta g\left(1-\frac{\kappa_3^2}{\kappa^2}\right) - (\alpha-\nu)^2\kappa^4\right] \div (12\pi s'^2) \qquad (58)$$

and

$$C_3' = \left[J_0\kappa^4\left(1-\frac{\kappa_3^2}{\kappa^2}\right)(\alpha-\nu)\right] \div (12\pi^2 s') \qquad (59)$$

Finally, solution of equation 44 gives

$$\varphi_{ii} = \left[\frac{\varphi_{33}}{1-(\kappa_3^2/\kappa^2)}\right] + \left(\frac{J_0\kappa^2}{12\pi^2}\right)e^{-2\nu\kappa^2(t-t_0)} \qquad (60)$$

Although the quantities φ_{ij}, γ_i, and δ are of interest in themselves, it is somewhat easier to interpret quantities that have been integrated over all directions in wave-number space as suggested by *Batchelor* [1953]. Thus, a quantity ψ_{ij} can be defined by the equation

$$\psi_{ij}(\kappa) = \int_0^A \varphi_{ij}\, dA \qquad (61)$$

where A is the area of a sphere of radius κ.

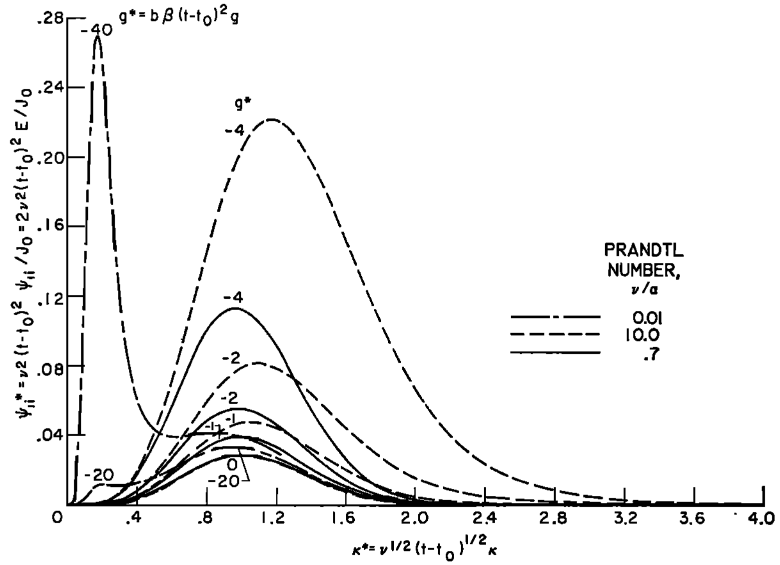

Fig. 2*a*. Dimensionless spectra of $\overline{u_i u_i}$ (energy spectra) with buoyancy forces destabilizing.

Then, since

$$\overline{u_i u_j} = \int_0^\infty \psi_{ij}\, d\kappa \tag{62}$$

(let $r = 0$ in equation 20), $\psi_{ij}\, d\kappa$ gives the contribution from the wave-number band $d\kappa$ to $\overline{u_i u_j}$.

The equations for φ_{33}, φ_{ii}, γ_3, and δ can be written in spherical coordinates by using the transformations

$$\kappa_1 = \kappa \cos\varphi \sin\theta$$
$$\kappa_2 = \kappa \sin\varphi \sin\theta$$
$$\kappa_3 = \kappa \cos\theta$$

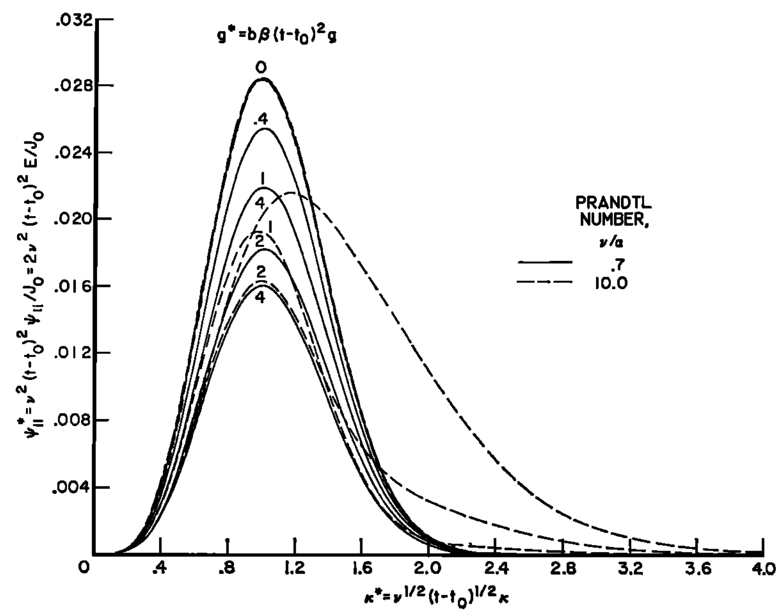

Fig. 2*b*. Dimensionless spectra of $\overline{u_i u_i}$ (energy spectra) with buoyancy forces stabilizing

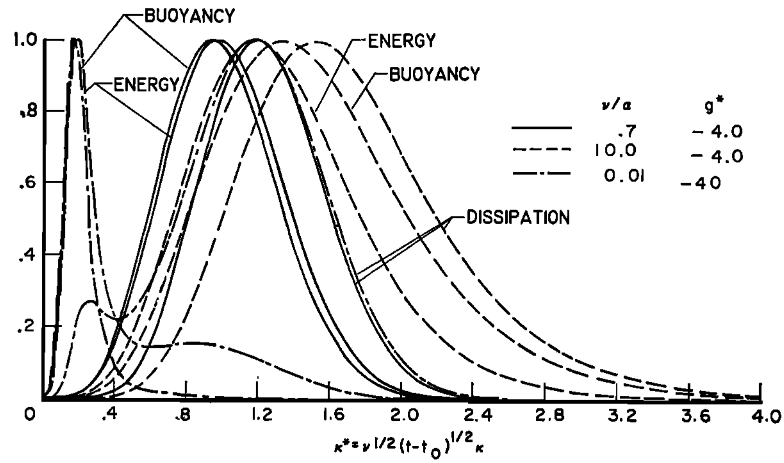

Fig. 3. Comparison of normalized energy, dissipation, and buoyancy spectra with buoyancy forces destabilizing.

Then, since φ_{33} (as well as φ_{ii}, γ_3, and δ) is not a function of the angle φ, the expression for ψ_{33} from equation 61 can be written as

$$\psi_{33} = 4\pi\kappa^2 \int_0^1 \varphi_{33}\, d(\cos\theta) \qquad (63)$$

We can write similar expressions for ψ_{ii}, γ_3, and δ integrated over all directions in wave-number space:

$$\psi_{ii} = 4\pi\kappa^2 \int_0^1 \varphi_{ii}\, d(\cos\theta) \qquad (64)$$

$$\Gamma_3 = 4\pi\kappa^2 \int_0^1 \gamma_3\, d(\cos\theta) \qquad (65)$$

$$\Delta = 4\pi\kappa^2 \int_0^1 \delta\, d(\cos\theta) \qquad (66)$$

Letting $r = 0$ in equations 25 and 27,

$$\overline{\tau u_j} = \int_0^\infty \Gamma_j\, d\kappa \qquad (67)$$

and

$$\overline{\tau^2} = \int_0^\infty \Delta\, d\kappa \qquad (68)$$

so that, as in the case of ψ_{ii}, $\Gamma_j\, d\kappa$ and $\Delta\, d\kappa$ give, respectively, contributions from the wave-number band $d\kappa$ to $\overline{\tau u_j}$ and $\overline{\tau^2}$. Computed spectra of the various turbulent quantities will be considered in the next section.

Results and Discussion

Before we consider in detail the spectra computed from the foregoing analysis, it may be worth while to indicate physically how the buoyancy forces would be expected to alter the turbulence. Figure 1 shows the effects of a negative and a positive vertical temperature gradient with the body force directed downward. For a negative temperature gradient, a turbulent eddy moving upward, for instance, will usually be hotter than the surrounding fluid. If the fluid has a positive temperature-expansion coefficient, the eddy will also be less dense than the surrounding fluid, so that buoyancy forces will tend to accelerate it upward. Similarly, an eddy moving downward will usually be accelerated downward. Thus, the negative temperature gradient tends to feed energy into the turbulent field, so that its effect is destabilizing. For a positive temperature gradient, it can be seen that the effect will be opposite to that just described; that is, the buoyancy forces will tend to stabilize the fluid.

Dimensionless energy spectra (spectra of $\overline{u_i u_i}$) are plotted in Figure 2. For making the calculations, the indicated integration in equation 64 was carried out numerically. When plotted, using the similarity variables shown, the spectrum for no buoyancy forces ($g^* = 0$) does not change with time, so that comparison of the various curves indicates how buoyancy effects will alter the spectrum. Thus, if a dimensionless spectrum curve lies above the curve for $g^* = 0$, the turbulent energy for that case is greater than it would be for no buoyancy forces. The turbulence itself is, of course, decaying with time. Curves are shown for Prandtl numbers ν/α of

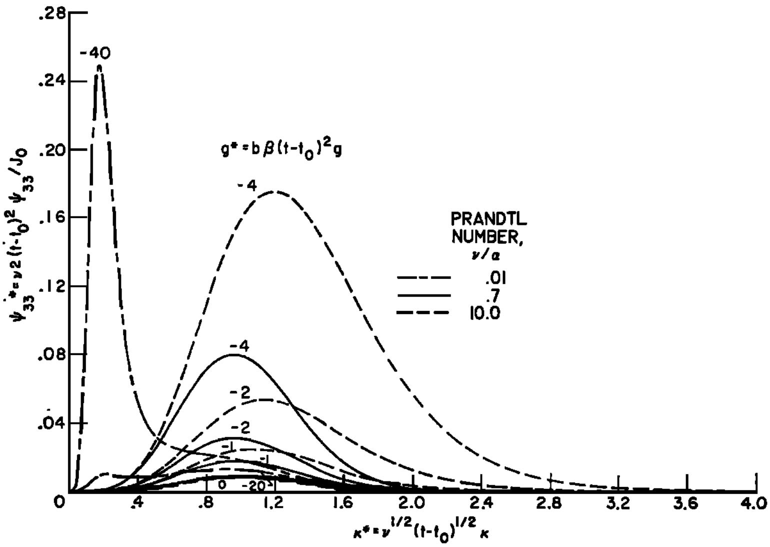

Fig. 4*a*. Dimensionless spectra of $\overline{u_3^2}$ with buoyancy forces destabilizing.

0.7, 10, and 0.01. These Prandtl numbers correspond, respectively, as far as order of magnitude is concerned, to a gas, a liquid like water, and a liquid metal.

Negative values of the buoyancy parameter g^*, defined as $b\beta(t - t_0)^2g$, correspond to negative temperature gradients, and positive values correspond to positive temperature gradients. (The quantity b in the definition of g^* is the temperature gradient.) In agreement with the discussion in connection with Figure 1, the areas under the spectrum curves increase for negative temperature gradients and, in general, decrease for positive ones. A reversal of the expected trend is shown by the curve for a Prandtl number of 10 and a g^* of 4. The action of the buoyancy forces in producing turbulent energy is particularly evident for a Prandtl number of 0.01 and negative values of g^*. There, the buoyancy forces tend to produce an extra peak in the spectra in the low-wave-number or large-eddy region.

Terms in the spectral energy equation, as well as energy spectra, are plotted in Figure 3 for cases in which the buoyancy forces augment the

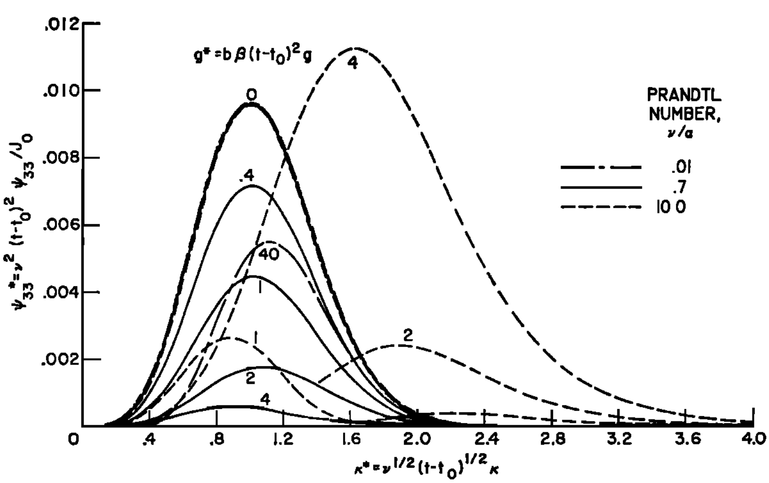

Fig. 4*b*. Dimensionless spectra of $\overline{u_3^2}$ with buoyancy forces stabilizing.

turbulence. The curves are normalized to the same height for comparison. The terms for the energy equation were obtained by integrating the terms in equation 44 over all directions in wave-number space by using equations 64 and 65. The second term in equation 44 gives the turbulent dissipation, and the last term gives the effect of buoyancy forces on the turbulence.

Consider first the curves in Figure 3 for Prandtl numbers less than 1. Those curves indicate that the spectrum of the buoyancy term tends to coincide with the energy spectrum for Prandtl numbers less than 1. That is, the energy from the buoyancy forces feeds into most of the parts of the energy spectrum. On the other hand, the dissipation regions are considerably separated from the energy-containing regions, the separation being greater for the lower Prandtl number. The dissipation regions for the two Prandtl numbers are close together, thus indicating that buoyancy forces, which are influenced by Prandtl number, do not greatly influence the dissipation for Prandtl numbers less than 1. The dissipation occurs mostly at high wave numbers, where the effect of buoyancy forces is not important. The low-wave-number parts of the energy spectrum, by contrast, are much more affected by buoyancy forces at low Prandtl numbers than at higher ones, because the eddies associated with the temperature-velocity correlations (see equation 7) are much larger at low Prandtl numbers. The spectra of the temperature-velocity correlations will be considered later (see Fig. 7).

The curves in Figure 3 for a Prandtl number of 10 indicate that for high Prandtl numbers, in contrast to the case of Prandtl numbers less than 1, the buoyancy forces can act on the small eddies. As a result of this effect, the buoyancy forces alter the dissipation spectrum for high-Prandtl-number fluids.

Dimensionless spectra of $\overline{u_3^2}$, which is the component of the turbulent energy in the direction of the temperature gradient and body force, are presented in Figure 4. The curves are somewhat similar to those for the spectra of $\overline{u_i u_i}$ and exhibit double peaks at the low Prandtl number. However, some of the spectra for $\overline{u_3^2}$ also have double peaks for a Prandtl number of 10. These are apparently caused by the action of the buoyancy forces on the small eddies. Another unexpected result is that the curve for a Prandtl number of 10 and a g^* of 4, although for a case where the buoyancy forces would be expected to be stabilizing, lies above the curve for no buoyancy effects. The physical reason for this result is not yet clear. It may be that some of the eddies, in this case, oscillate several times before being damped out.

In general, the turbulence is anisotropic. The anisotropy of the turbulence is clearly seen in Figure 5, where the spectrum curves for $\overline{u_3^2}$

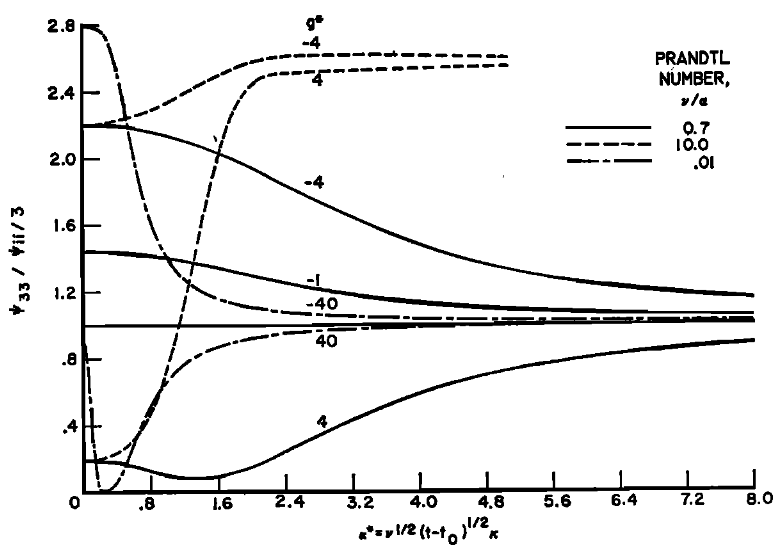

Fig. 5. Curves showing ratio of spectrum curves for $\overline{u_3^2}$ to those for $\overline{u_i u_i}/3$.

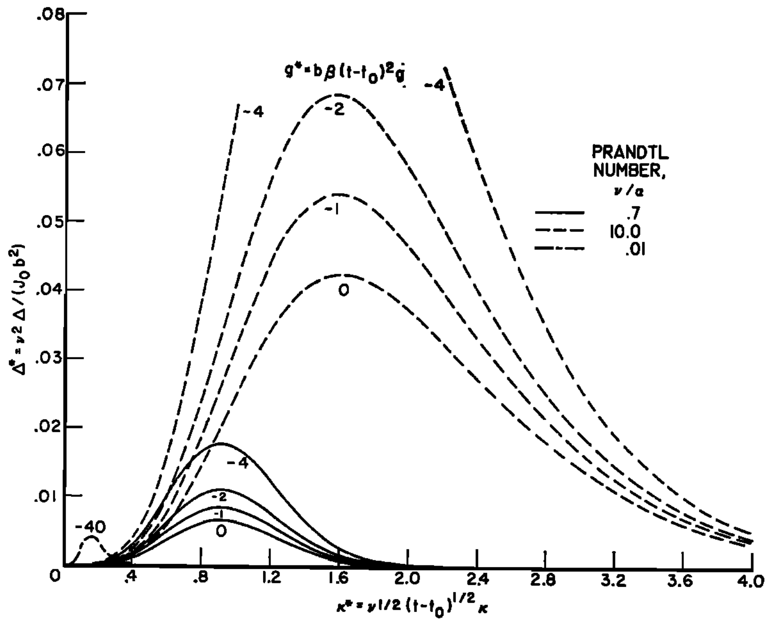

Fig. 6*a*. Dimensionless spectra of temperature variance $\overline{\tau^2}$ with buoyancy forces destabilizing.

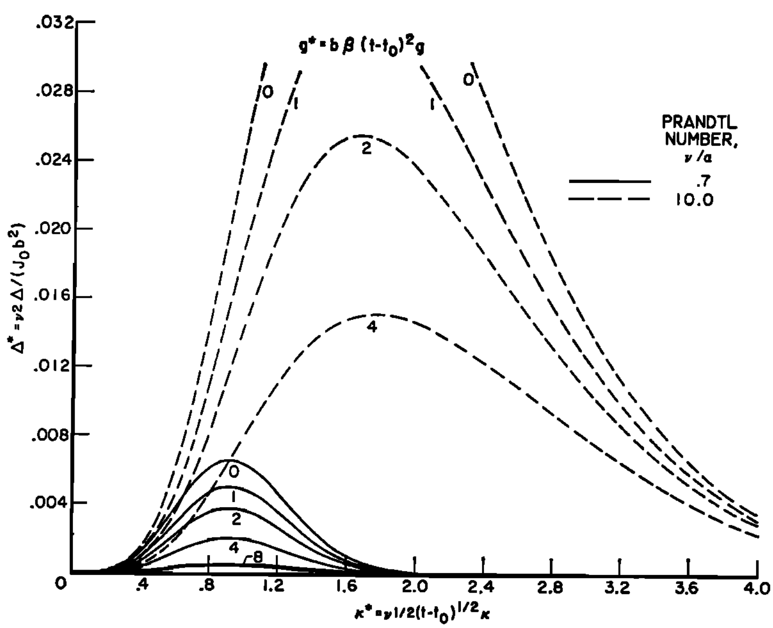

Fig. 6*b*. Dimensionless spectra of temperature variance $\overline{\tau^2}$ with buoyancy forces stabilizing.

divided by those for $\overline{u_i u_i}/3$ are plotted. For isotropic turbulence all values of $\psi_{33}/(\psi_{ii}/3)$ would be 1, inasmuch as $\psi_{ii}/3$ represents the average spectrum of the components of the energy. For destabilizing conditions $\overline{u_3^2}$ is higher than the average component, whereas for stabilizing conditions it is lower. This is physically reasonable, inasmuch as the buoyancy forces would be expected to act mainly on the vertical components of the velocities of the eddies. In fact, equation 36 indicates that the buoyancy terms (last two terms) occur only in the equation for φ_{33} for a vertical body force.

For Prandtl numbers less than 1 the anisotropy is most pronounced in the large-eddy region, so that apparently the buoyancy forces act mostly on the large eddies. In the small-eddy region the curves for Prandtl numbers less than 1 approach 1, so that the turbulence is isotropic in the smallest eddies. Thus, the theory of local isotropy seems to apply here. This observation is in opposition to that for weak turbulence with a uniform velocity gradient, where local isotropy was absent [*Deissler*, 1961*a*]. Also, the curves in Figure 5 for a Prandtl number greater than 1 do not show local isotropy. Thus, local isotropy seems to be obtained only for Prandtl numbers less than 1 in the present analysis. The situation may be different for high Reynolds numbers.

It was originally thought that the difference between the results for Prandtl numbers less than, and greater than, 1 was caused by a difference in the effect of pressure forces in the two cases. A calculation with the pressure force terms absent, however, indicated that those terms have but a minor effect on the results. It appears that the effect is due to the way the buoyancy forces act in the two cases and that the buoyancy forces can act on the smaller eddies at high Prandtl numbers. This is in agreement with the curves in Figure 3.

Spectra of the temperature variance $\overline{\tau^2}$ are plotted in Figure 6. For $g^* = 0$, the results reduce to those of *Dunn and Reid* [1958]. The trends with g^* are similar to those for the spectra of $\overline{u_i u_i}$; that is, the areas under the curves are larger for negative than for positive temperature gradients. However, the areas under the curves for low Prandtl numbers are much smaller than for the higher ones because, for the same viscosity, the high thermal conductivities associated with lower-Prandtl-number fluids tend to smear out the temperature fluctuations. As Prandtl number decreases, the spectra move into the lower-wave-number regions because the conduction effects tend to destroy the small temperature eddies more readily than larger ones.

The last spectra to be considered are those of the temperature-velocity correlations $\overline{\tau u_3}$. These are plotted in dimensionless form in Figure 7. The quantity $\overline{\tau u_3}$ is proportional to the turbulent heat transfer. The total heat transfer q_3 is the sum of the laminar and turbulent heat transfer; it is given by

$$q_3 = -k(dT/dx_3) + \rho c_p \overline{\tau u_3}$$

where k is the thermal conductivity and c_p is the specific heat at constant pressure. Inasmuch as the temperature gradient b occurs in the denominator of the dimensionless spectrum function in Figure 7, those curves can also be considered as the spectra of the eddy diffusivity for heat transfer. The eddy diffusivity for heat transfer ϵ_h is defined by

$$\epsilon_h = -\frac{\overline{\tau u_3}}{dT/dx_3}$$

The spectra indicate that, when the buoyancy forces are destabilizing, the turbulent heat transfer is greater than it would be without buoyancy effects. This is congruous with the effect of buoyancy forces on the turbulent intensity shown in Figure 2. Similarly, for positive values of g^*, the turbulent heat transfer is less than it would be for no buoyancy forces. However, as g^* continues to increase, the turbulent heat transfer goes to zero and then changes sign. That is, the turbulence begins to transfer heat against the temperature gradient. This is shown somewhat more clearly in Figure 8, where the temperature-velocity correlation coefficient $\overline{\tau u_3}/[(\overline{\tau^2})^{1/2}(\overline{u^2})^{1/2}]$ is plotted against g^*. As g^* increases, the sign of the turbulent heat transfer oscillates. Although these are rather surprising results, turbulence has on occasion been observed to pump heat against a temperature gradient. This occurs, for instance, in a Ranque-Hilsch vortex tube, where expansion and contraction of eddies in a pressure gradient can cause heat to flow against a temperature gradient. The effect observed here, however, appears to be caused by the action of the buoy-

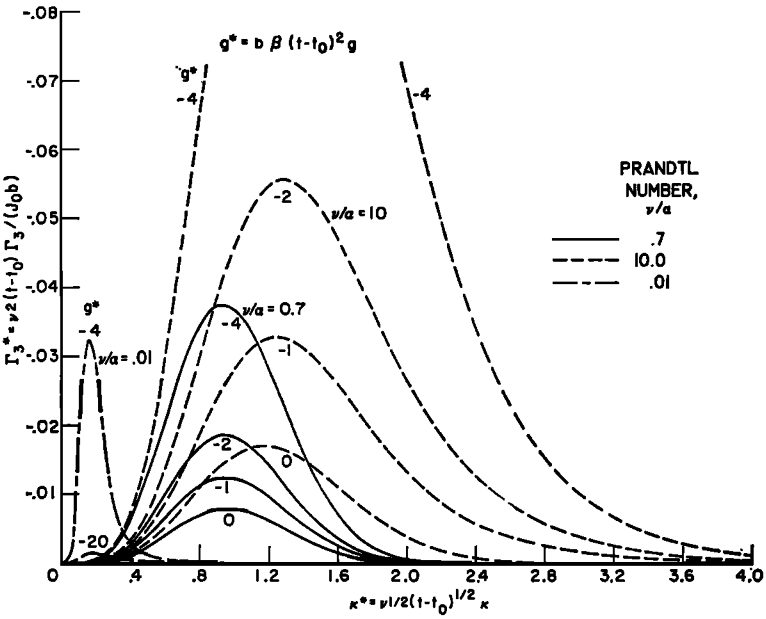

Fig. 7a. Dimensionless spectra of temperature-velocity correlations $\overline{\tau u_3}$ with buoyancy forces destabilizing.

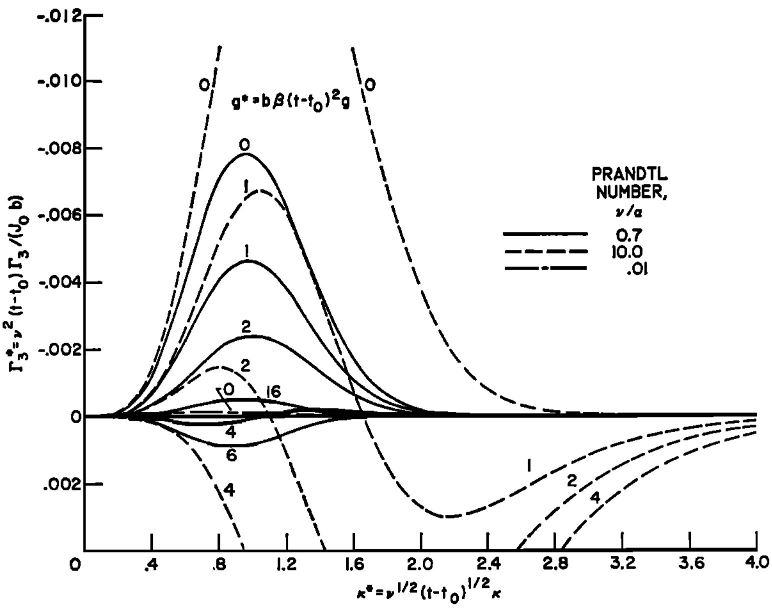

Fig. 7b. Dimensionless spectra of temperature-velocity correlations $\overline{\tau u_3}$ with buoyancy forces stabilizing.

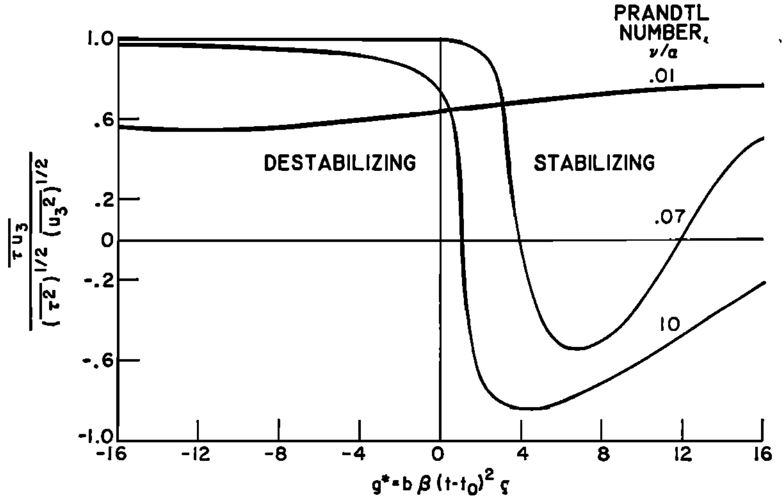

Fig. 8. Temperature-velocity correlation coefficient as a function of buoyancy parameter.

ancy forces on the eddies. In the stabilizing case, the buoyancy forces ordinarily act in the direction opposite to that in which an eddy starts to move (see Fig. 1), and so the sign of the velocity fluctuation might be changed without necessarily changing the sign of the corresponding temperature fluctuation. Thus, it appears possible that the direction of the turbulent heat transfer could be reversed.

For negative values of g^*, Figure 8 indicates that nearly perfect correlation between the temperature and velocity fluctuations is approached. This, again, can be explained by the action of the buoyancy forces. Thus, as was mentioned previously, an eddy moving upward in a negative temperature gradient will usually be hotter than the surrounding fluid and so will be pushed upward still more by the buoyancy forces. If an eddy moving upward happens to be cooler than the surrounding fluid, it will be pushed downward. Therefore, positive contributions to $\overline{\tau u_3'}$ are amplified, whereas negative contributions are damped out by the buoyancy forces, so that the net effect is to increase the value of $\overline{\tau u_3'}$ toward 1.

Conclusion

It appears that by using the present method of analysis—that is, by neglecting triple correlations and limiting the investigation to a reasonably weak turbulence—we can profitably study many of the turbulent processes. It is true that because we neglected triple correlations we were not able to study the transfer of energy between eddies of various sizes, but that is only one of the important processes occurring in turbulence and can be studied separately. For instance, we could, like *Deissler* [1960], consider three- and four-point correlation equations and neglect fifth-order correlations. However, if that were done in the present case, where buoyancy effects are considered, the problem might tend to get out of hand. Alternatively, if a mean velocity gradient as well as a temperature gradient were included, we would obtain a transfer of energy from large to small eddies, like *Deissler* [1961*a*], even though triple correlations were neglected. It appears that the method of analysis followed here gives information about other turbulent processes such as the dissipation and the production or extraction of energy by buoyancy forces.

It might be emphasized that no steady-state solution was obtained here. Although the buoyancy forces could produce turbulent energy, the production was never quite great enough to offset the dissipation. It appears that, in order to obtain a steady-state solution, we would have to consider a mean temperature profile with higher-order derivatives, rather than a linear one, as has been considered here. The same observation applies to a turbulent shear flow, where no steady-state solution was obtained for a linear velocity profile [*Deissler*, 1961*a*]. The implications of these points for turbulence in meteorology and oceanography would seem to be that a Richardson number containing only first spatial derivatives of the mean velocity and temperature are probably not adequate for specifying a steady-state turbulent field. It may also be necessary to specify at least the second derivatives of those quantities.

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