Thermal Turbulence at Very Small Prandtl Number[1]

EDWARD A. SPIEGEL

Courant Institute of Mathematical Sciences
New York University, New York

Abstract. The equations of thermal turbulence are derived for the case of small Prandtl number, a case of a relatively simple realizable turbulent flow. The power spectrum of velocity is described using the transfer functions of Heisenberg and of Kovasznay. It is seen that these do not give uniformly good approximations, since they force spectral energy to flow from low to high wave numbers, though the Kovasznay approximation may be useful for large Rayleigh numbers. A general comparison of the present study with the Malkus theory of convection indicates disagreement which probably results from the dominance of nonlinear terms in the low-Prandtl-number limit.

Introduction. In treating the dynamics of thermal turbulence one is confronted with two kinds of nonlinearity. The first kind results from distortions of the mean temperature profile; the second describes the self-interaction of the turbulent velocity field and the interaction of the velocity field with the random component of the temperature field. The fluctuation interactions, as we shall call them, appear as bilinear terms in the equations of motion and give rise to the well-known closure problem of turbulence. They have been studied with only modest success in the theory of homogeneous turbulence.

In his attack on the thermal turbulence problem, *Malkus* [1954*b*] has suggested that, rather than treat the fluctuation interactions explicitly, one should adopt some hypothesis that implicitly includes their net effect. In the Malkus theory, this hypothesis is a maximization principle for statistically steady turbulence; for example, in the work cited Malkus maximizes the heat transported by the fluid. To actually carry out such a maximization Malkus has introduced an interesting formalism based on some additional assumptions. None of the individual assumptions of the theory has been experimentally tested, but its predictions have been in good agreement with the available experimental data [*Townsend*, 1962].

If there is a preferred state of statistically steady turbulence, it seems reasonable to suppose that this may be inferred from a theory based on a physically meaningful approximation to the fluctuation interactions. The situation may be analogous to that in microscopic statistical mechanics, where one hopes to deduce the canonical distribution from the Boltzmann equation, using an approximate representation of molecular interactions.

In the present paper we shall describe and amplify an approach to the study of thermal turbulence suggested by *Ledoux, Schwarzschild, and Spiegel* [1961] (this paper will hereinafter be referred to as I) in which the fluctuation interactions are treated by approximations suggested in theories of homogeneous turbulence. The work outlined here will be restricted to the case of very small Prandtl number. Since the calculations are still in progress, the approach rather than the results will be stressed. As we shall see, the approach can lead to information about velocity and temperature spectra which does not seem to be readily deducible from the Malkus theory. An explicit comparison of the two approaches cannot be made properly as yet, though this possibility should occur in later treatments.

The equations of the problem. We shall restrict ourselves to convection between two horizontal plates at fixed constant temperatures. The lower plate is at $z = 0$ and at a temperature ΔT higher than the plate at $z = d$. The Boussinesq approximation will be adopted as well as the so-called free-boundary conditions. The general equations for this situation exist in many places in the literature (e.g., *Malkus and Veronis* [1958].

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Denoting a horizontal average by an overbar, let us write the total temperature as

$$T(x, y, z; t) = \bar{T}(z) + \theta(x, y, z; t) \tag{1}$$

where θ is the fluctuating part of the temperature and $\bar{\theta} = 0$. It follows from the heat equation that in steady turbulent convection the kinematic heat transport is constant, and given by

$$H = \overline{w\theta} + \kappa\beta \tag{2}$$

where

$$\beta = -d\bar{T}/dz \tag{3}$$

w is the vertical component of velocity, and κ is the thermometric conductivity. In the absence of motion β must be constant, but when convective transport occurs β varies with z, being larger at the boundaries where w must vanish.

The equations for the fluctuating quantities are

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\nabla^2\mathbf{u} - g\alpha\theta\boldsymbol{\lambda} + \frac{1}{\rho}\Delta p = -\mathbf{u}\cdot\nabla\mathbf{u} \tag{4}$$

and

$$\frac{\partial \theta}{\partial t} - \kappa\nabla^2\theta + \beta w = -\mathbf{u}\cdot\nabla\theta + \overline{\mathbf{u}\cdot\nabla\theta} \tag{5}$$

where p is the deviation of the pressure from its horizontal average and $\boldsymbol{\lambda}$ is a vertical unit vector. To these equations we may add the incompressibility condition

$$\nabla\cdot\mathbf{u} = 0 \tag{6}$$

Equations 4 and 5 have been written with the fluctuation interactions on the right-hand sides; these are the nonlinear terms we intend to approximate. The term βw is also nonlinear, though it does not appear so explicitly.

As was mentioned above, we shall limit ourselves to very small Prandtl numbers. To see the meaning of this restriction, let us introduce d, d^2/ν, ν/d, and $\kappa\nu/g\alpha d^3$ as the units of length, time, velocity, and temperature, respectively. Equations 2, 4, and 5 then become

$$\left(\frac{d}{\kappa\,\Delta T}\right)H = \frac{\sigma}{R}\,\overline{w\theta} + \frac{d}{\Delta T}\,\beta \tag{2a}$$

$$\frac{\partial \mathbf{u}}{\partial t} - \nabla^2\mathbf{u} + \Pi\frac{1}{\rho}\nabla p - \frac{1}{\sigma}\,\theta\boldsymbol{\lambda} = -\mathbf{u}\cdot\nabla\mathbf{u} \tag{4a}$$

$$\sigma\frac{\partial \theta}{\partial t} - \nabla^2\theta + \sigma R w\frac{\beta d}{\Delta T} = -\sigma(\mathbf{u}\cdot\nabla\theta - \overline{\mathbf{u}\cdot\nabla\theta}) \tag{5a}$$

where

$$\sigma = \nu/\kappa \tag{7}$$

$$R = g\alpha\,\Delta T\,d^3/\kappa\nu \tag{8}$$

and Π is a nondimensional standard pressure.

Considering now the case of small Prandtl number, σ, let us expand $\mathbf{u}$ and θ in Taylor series:

$$\left.\begin{aligned}\mathbf{u} &= \mathbf{u}_0 + \sigma\mathbf{u}_1 + \cdots\\ \theta &= \theta_0 + \sigma\theta_1 + \cdots\end{aligned}\right\} \tag{9}$$

If we then introduce these series into equation 4a we find

$$\theta_0 = 0 \tag{10}$$

and

$$\frac{\partial \mathbf{u}_0}{\partial t} - \nabla^2\mathbf{u}_0 + \Pi\frac{1}{\rho}\nabla p_0 - \theta_1\boldsymbol{\lambda} = -\mathbf{u}_0\cdot\nabla\mathbf{u}_0 \tag{11}$$

Thus, we see that θ is linear with σ for small σ. Making use of this knowledge we learn from equation 2a that, to leading order in σ,

$$H\left(\frac{d}{\kappa\,\Delta T}\right) = \frac{\beta d}{\Delta T} + 0(\sigma^2) \tag{12}$$

that is, β is constant and convective heat transfer is much less than conductive transfer. Further, since ΔT is fixed,

$$\beta = \Delta T/d \tag{13}$$

which simplifies equation 5a. With this simplification, equation 5a to leading order is

$$\nabla^2\theta_1 = -Rw_0 \tag{14}$$

The simple form of equation 14 makes the case of low Prandtl number very attractive, since we do not have to deal with turbulent conductivity at all.

Finally, let us specify the boundary conditions of the problem. In the nondimensional coordinates the fluid is bounded by planes at $z = 0$ and 1. We adopt the so-called free-boundary conditions, namely,

$$w = \frac{\partial^2 w}{\partial z^2} = \theta = 0 \quad \text{at} \quad z = 0, 1 \tag{15}$$

We will also assume that the layer has a square horizontal cross section of dimension L, where L is to be considered arbitrarily large.

The spectral equations. In the procedure discussed in I, we expand the velocity and temperature fields in an appropriate complete set. It is suggested that the normal modes of the linearized equations, with β given its correct form, make up the relevant set. In the present case of low Prandtl number, these normal modes are combinations of sines and cosines. Thus, if we give due attention to boundary conditions, w and θ should be expanded as follows:

$$w = \sum_{\mathbf{a}} \sum_{n=1}^{\infty} W_{n\mathbf{a}} e^{i\mathbf{a}\cdot\mathbf{x}} \sin n\pi z \qquad (16)$$

and

$$\theta = \sum_{\mathbf{a}} \sum_{n=1}^{\infty} \Theta_{n\mathbf{a}} e^{i\mathbf{a}\cdot\mathbf{x}} \sin n\pi z \qquad (17)$$

Here n is an integer, $\mathbf{a}$ is the horizontal wave vector with components k_x and k_y, and the summations extend over all positive n and all k_x and k_y that are integral multiples of π/L. It should be remembered that all quantities are nondimensional.

Corresponding expansions for u and v may be written. If we first take account of the incompressibility condition (6), these are seen to be

$$u = \sum_{\mathbf{a}} \sum_{n=1}^{\infty} U_{n\mathbf{a}} e^{i\mathbf{a}\cdot\mathbf{x}} \cos n\pi z \qquad (18)$$

and

$$v = \sum_{\mathbf{a}} \sum_{n=1}^{\infty} V_{n\mathbf{a}} e^{i\mathbf{a}\cdot\mathbf{x}} \cos n\pi z \qquad (19)$$

with the auxiliary condition

$$n\pi W_{n\mathbf{a}} + ik_x U_{n\mathbf{a}} + ik_y V_{n\mathbf{a}} = 0 \qquad (20)$$

Similarly we have

$$\frac{p}{\rho} = \sum_{\mathbf{a}} \sum_{n=1}^{\infty} P_{n\mathbf{a}} e^{i\mathbf{a}\cdot\mathbf{x}} \cos n\pi z \qquad (21)$$

Now, on making use of equation 14 we have

$$k^2 \Theta_{n\mathbf{a}} = R W_{n\mathbf{a}} \qquad (22)$$

where

$$k^2 = n^2\pi^2 + k_x^2 + k_y^2 = n^2\pi^2 + a^2 \qquad (23)$$

Using the above expansions, conditions 17 and 20, and the statement $\partial/\partial t \langle\ \rangle = 0$, where $\langle\ \rangle$ denotes ensemble average, we then find

$$\left.\begin{aligned} 2(k^2 - R/k^2)\langle W_{n\mathbf{a}} W_{n\mathbf{a}}^* \rangle &= (NL)_z \\ 2k^2 \langle U_{n\mathbf{a}} U_{n\mathbf{a}}^* \rangle &= (NL)_x \\ 2k^2 \langle V_{n\mathbf{a}} V_{n\mathbf{a}}^* \rangle &= (NL)_y \end{aligned}\right\} \qquad (24)$$

The various terms designated by NL are complicated trilinear terms in velocity components familiar in the theory of homogeneous turbulence [e.g., *Batchelor*, 1953]. We shall not write them out explicitly, since these are the terms we shall approximate.

Equations 24 show the anisotropic character of the problem. Only one approximation has been proposed for the NL terms applicable to such a case: the direct-interaction approximation of *Kraichnan* [1959], and it is probably the only approximation now available that can lead to a fully consistent picture. However, the NL expression that results from Kraichnan's theory is extremely difficult to employ, and we have therefore elected to begin by introducing simpler approximations to NL devised for isotropic turbulence. To use these approximations we need to simplify our equations further.

First, instead of dealing with three simultaneous equations for the kinetic energies, we write a single equation for the three-dimensional energy spectrum, F, defined by

$$F(\mathbf{k}) = \frac{L^2}{8\pi^3 d^2} k^2 \langle U_{n\mathbf{a}} U_{n\mathbf{a}}^* + V_{n\mathbf{a}} V_{n\mathbf{a}}^* + W_{n\mathbf{a}} W_{n\mathbf{a}}^* \rangle \qquad (25)$$

From equations 24 we obtain

$$2k^2 F(\mathbf{k}) - \frac{2R}{k^2} \frac{L^2}{8\pi^3 d^2} k^2 \langle W_{n\mathbf{a}} W_{n\mathbf{a}}^* \rangle = T(\mathbf{k}) \qquad (26)$$

where $T(\mathbf{k})$ is the usual transfer function. We can express $\langle W_{n\mathbf{a}} W_{n\mathbf{a}}^* \rangle$ in terms of F if we assume (as outlined above) that the normal modes of the linear equation form a complete set and that the purely viscous modes are not important in the dynamics. (These viscous modes are horizontal eddies which may be excited by nonlinear interactions and which transport no heat; see I.) Then equation 26 becomes

$$2\eta(\mathbf{k}) F(\mathbf{k}) = -T(\mathbf{k}) \qquad (27)$$

where

$$\eta(\mathbf{k}) = (Ra^2/k^4) - k^2 \qquad (28)$$

The quantity η is the growth rate of a disturbance in the absence of nonlinear interactions (see I).

We are still not in a position to use the standard isotropic approximations for $T(\mathbf{k})$, since these give only estimates of the energy exchanged among modes of given absolute value of $\mathbf{k}$. Let us therefore integrate equation 27 over all angles in $\mathbf{k}$ space. We find

$$2\eta(k)F(k) = -T(k) \qquad (29)$$

where

$$F(k) = \int F(\mathbf{k})\, d\Omega \qquad (30)$$

$$T(k) = \int T(\mathbf{k})\, d\Omega \qquad (31)$$

and

$$\eta(k) = \frac{\int_0^{\pi/2} g_k(\vartheta)\eta(\mathbf{k}) \sin\vartheta\, d\vartheta}{\int_0^{\pi/2} g_k(\vartheta) \sin\vartheta\, d\vartheta} \qquad (32)$$

Here $g_k(\vartheta)$ is an appropriate weight factor.

Now (29) is rigorously true if $g_k(\vartheta) = F(\mathbf{k})$; but we do not yet know $F(\mathbf{k})$. Hence we must choose a first approximation for $g_k(\vartheta)$. In the absence of any other information, we take $g_k(\vartheta)$ as the density of states in $\mathbf{k}$ space. The situation is illustrated in Figure 1, which shows the k_xk_z plane. The boundary conditions permit only the values of k that fall on the planes $k_z = n\pi$, $n = 1, 2, 3, \cdots$. The horizontal lines are projections of these planes on the k_xk_z plane. Then function g is

$$g_k(\vartheta) = \frac{1}{\cos\vartheta} \sum_{m=1}^{[k/\pi]} \delta\left(\vartheta - \cos^{-1}\frac{m\pi}{k}\right) \qquad (33)$$

where δ is the Dirac function. For η we then obtain

$$\eta(k) = \frac{R}{k^2} \cdot \left[1 - \frac{\pi^2}{k^2}\left(\sum_{m=1}^{[k/\pi]} m \Big/ \sum_{m=1}^{[k/\pi]} \frac{1}{m}\right)\right] - k^2 \qquad (34)$$

Thus, η is a discontinuous function of k, the discontinuities occurring at the onset of each new vertical mode. These discontinuities may possibly be connected with the discrete transitions observed and discussed by *Malkus* [1954*a*]. However, we shall here smooth them over and take for η the expression

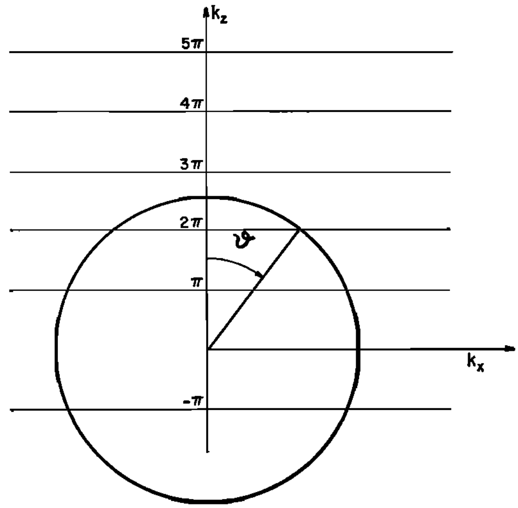

Fig. 1. Illustrating the averaging over angle in $\mathbf{k}$ space.

$$\eta(k) = [R(k^2 - \pi^2)/k^4] - k^2 \qquad (35)$$

with the stipulation that k cannot be less than π, the smallest value allowed by the boundary conditions. This procedure of averaging $\eta(\mathbf{k})$ differs from that in I, where η was given its maximum value as a function of a for each n. The resulting expression in that case is

$$\eta_{\text{I}} = \left(\frac{R}{2k^2} - k^2\right) \qquad k \geq k_0 = \sqrt{2}\pi \quad (35a)$$

We also note that (35) is an entirely satisfactory approximation to (34), as may be seen in Figure 2, where the two expressions are compared for $R \to \infty$.

Our final approximation will be to treat k as continuous rather than discrete, retaining π as the smallest allowed value of k. We have already done so implicitly for $\mathbf{a}$, having imagined that $L \to \infty$. But in letting n be continuous we are making a significant approximation. This is necessary in order to utilize the approximations for $T(k)$ discussed in the next section.

The behavior of η as a function of k may now be outlined. For fixed R, $\eta = -\pi^2$ when $k = \pi$,

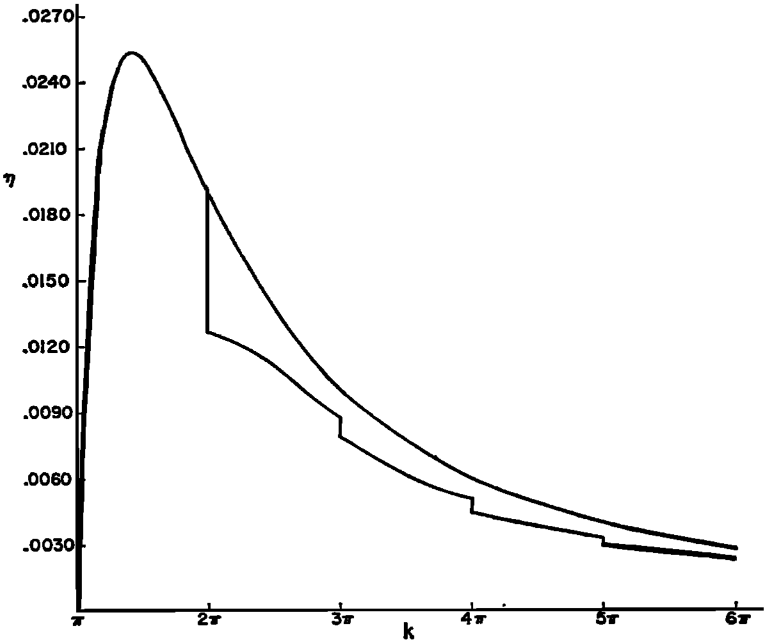

Fig. 2. η as a function of k for $R = \infty$. The broken curve represents equation 34.

the smallest allowed wave number. With increasing k, η rises to a maximum value and then falls monotonically, becoming negative once more when k is so large that dissipative processes become important. The value k_1 at which η first becomes zero with increasing k is never much greater than π. In Figure 3 we show the dependence of k_1 and of k_2, the wave number of maximum η, as functions of R.

Some approximate solutions. In this section we shall examine the consequences of adopting the approximations for the transfer function due to Heisenberg and Kovasznay. These are usually expressed for the definite integral of $T(k)$, and *Heisenberg's* [1948] approximation is

$$\int_\pi^k T(k')\,dk' = -\gamma \int_k^\infty \left[\frac{F(p)}{p^3}\right]^{1/2} dp \int_\pi^k q^2 F(q)\,dq \quad (36)$$

where γ is a nondimensional coupling constant.

Differentiation then yields

$$T(k) = -\gamma k^2 F(k) \int_k^\infty \left[\frac{F(p)}{p^3}\right]^{1/2} dp + \gamma\left[\frac{F(k)}{k^3}\right]^{1/2} \int_\pi^k q^2 F(q)\,dq \quad (37)$$

Now, since $\eta = 0$ at $k = k_1$, we have, by equation 29,

$$T(k_1) = 0 \quad (38)$$

whence

$$k_1^{7/2} \int_{k_1}^\infty \left[\frac{F(p)}{p^3}\right]^{1/2} dp = [F(k_1)]^{-1/2} \int_\pi^{k_1} q^2 F(q)\,dq \quad (39)$$

As $R \rightarrow \infty$, $k_1 \rightarrow \pi$, and the left-hand side of equation 39 approaches a nonzero, constant value. Hence, if the Heisenberg approximation is to be applicable, the right-hand side of equation 39 must also approach a nonzero limit.

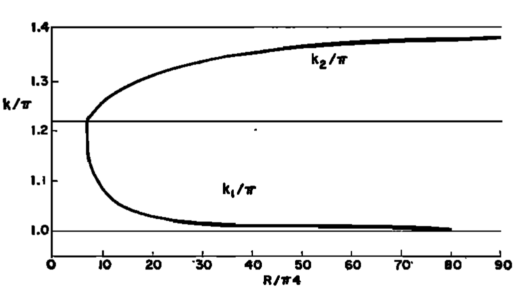

Fig. 3. k_1 and k_2 as functions of R/π^4.

To find the limit of the right-hand side of (39) for large R it suffices to express F in the Taylor series

$$F(k) = \sum_{n=1}^{\infty} \frac{1}{n!} F^{(n)}(\pi)(k-\pi)^n \tag{40}$$

On substituting this series into equation 39 and requiring that the right-hand side approach a constant as $k_1 \to \pi$ we find that

$$F \sim \text{constant}/(k-\pi)^2 \tag{41}$$

for $k \sim \pi$. Clearly, such behavior for the power spectrum is not admissible.

In I, where the Heisenberg approximation was used, the difficulty at $k = \pi$ was avoided by introducing an arbitrary cutoff in the input and spectrum at effectively $k = k_2$. Alternatively, we might consider trying a cutoff at $k = k_1$, but it is readily seen that this does not remove the difficulty. We must conclude that the Heisenberg approximation cannot be used over the full turbulent spectrum in this problem.

The difficulty in finding a physically meaningful spectrum with the Heisenberg approximation arises because this approximation always forces spectral energy from low to high wave numbers. When the input of energy has a maximum value away from the smallest wave number, energy should flow also to smaller wave numbers from the maximum. This is probably true in most real systems. Thus a transfer function is needed that permits energy to flow according to physical requirements. A simple function of this type has been devised by *Kraichnan and Spiegel* [1962], and the solution for the spectrum is being investigated. It is also of interest to consider yet another approximation, that of *Kovasznay* [1948].

In Kovasznay's approximation we have

$$\int_{\pi}^{k} T(k')\, dk' = -\gamma k^{5/2} F^{3/2}(k) \tag{42}$$

where γ has the same meaning, but not necessarily the same value, as before. Expression 42 leads to

$$T(k) = -5/2\gamma k^{3/2} F^{3/2}(k) - \tfrac{3}{2}\gamma k^{5/2}[F(k)]^{1/2}(dF/dk) \tag{43}$$

We see immediately that condition 38 may be satisfied in this approximation by requiring that

$$F(k_1) = 0 \tag{44}$$

If we then introduce the Kovasznay approximation into equation 29 we have no difficulty in solving for F. The solution is

$$F(k) = \frac{R^2}{64\gamma^2 k_1^7}\left(\frac{k}{k_1}\right)^{-5/3}\left\{\left[1-\left(\frac{k_1}{k}\right)^{8/3}\right] - \frac{4}{7}\left(\frac{\pi}{k_1}\right)^2\left[1-\left(\frac{k_1}{k}\right)^{14/3}\right] - \frac{2k_1^4}{R}\left[\left(\frac{k}{k_1}\right)^{4/3}-1\right]\right\}^2 \tag{45}$$

We see that F rises from the value zero at $k = k_1$ to a maximum and then descends to zero at some finite $k = k_*$. For $k > k_*$, F increases monatonically. In using the Kovasznay approximation in isotropic turbulence we usually take $F = 0$ for $k \geq k_*$, since k_* is normally large. Here, for large values of R, it is given by

$$k_* = \left\{\left[1 - \frac{4}{7}\left(\frac{\pi}{k_1}\right)^2\right]\frac{R}{4k_1^4}\right\}^{3/4} k_1 \tag{46}$$

Similarly, we see that F rises as k decreases from the value k_1, and we must impose $F = 0$ for $k \leq k_1$. Our solution is thus valid only for $k_1 \leq k \leq k_*$. However, for large R, $k_1 \simeq \pi$ and $k_* = \infty$, and if we restrict ourselves to this situation, the solution may have some qualitative value. Hence, letting

$$q = k/k_1 \tag{47}$$

we have the following reasonably simple expression for F:

$$F(q) = k_1\left(\frac{R}{8\gamma k_1^4}\right)^2 q^{-5/3}\left[(1-q^{-8/3}) - \frac{4}{7}(1-q^{-14/3}) - \frac{2k_1^4}{R}(q^{4/3}-1)\right]^2 \tag{48}$$

For $1 << q << \infty$, $F \propto q^{-5/3}$, so that the Kolmogoroff law is satisfied in a reasonable range of wave numbers. In Figure 4 we illustrate the form of the spectrum for several values of $R/2k_1^4$.

We can readily obtain the rms velocity at very large R for this approximation; it is, averaged over all space,

$$\langle v^2\rangle^{1/2} = \frac{R}{8\pi^4\gamma}\frac{\pi}{\sqrt{6}} = \frac{R}{6.2\gamma\pi^4} \tag{49}$$

where it should be recalled that ν/d is the unit of velocity. A comparison of this result with the

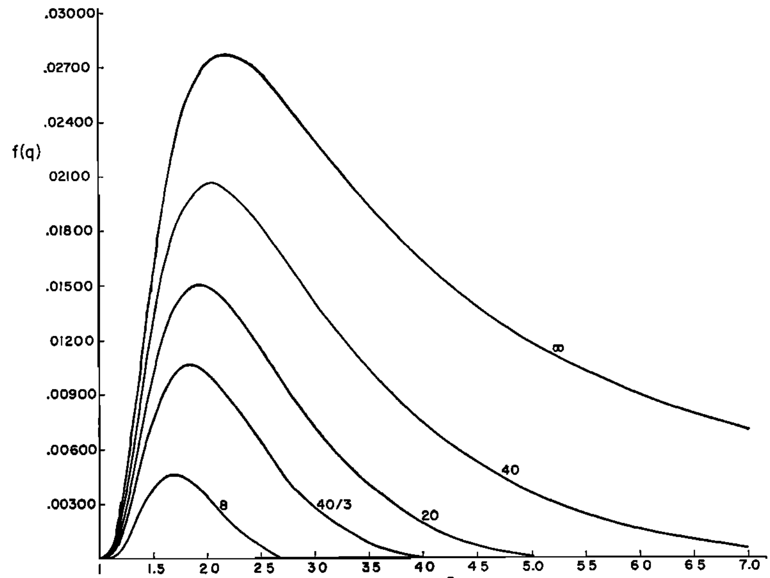

Fig. 4. $f = F/k_1/(R/8\gamma k_1^4)^2$ versus $q = k/k_1$ for different values of $R/2k_1^4$.

corresponding one obtained in I using the Heisenberg approximation is not too meaningful, since, as was discussed above, a modified input function was used there. The result given in I is 5 times larger than the present one, differences in the values of the coupling constants being neglected. Thus, amplitudes are very sensitive to approximations concerning the low-wave-number range, though the relative amplitudes of temperature and velocity are probably much less sensitive.

At this point we may also investigate the validity of our low-Prandtl-number equations. In particular, we must determine how small the Prandtl number must be to permit the neglect of the convection heat transfer term in equation 2. The calculation of the rms temperature fluctuation from relation 22 by means of the Kovasznay approximation together with the above results leads to the criterion

$$\sigma \ll \frac{1}{10} \frac{8\gamma\pi^4}{R} \tag{50}$$

for the neglect of convective heat transfer. When this criterion is met, the temperature gradient is sensibly constant and our simplified equations hold.

Concluding remarks. It cannot be said that the calculations presented here give very reliable quantitative information about thermal turbulence. However, they serve to illustrate several aspects of the problem that we should like to stress.

The first point of interest is the simplicity of equations 11 and 14, which govern thermal turbulence at very small Prandtl number. These equations describe a real physical system but still come very close to being the equations of homogeneous turbulence. Very small Prandtl numbers cannot yet be achieved experimentally, but in stellar atmospheres Prandtl numbers of the order of 10^{-9} do occur. Moreover, in the atmospheres of B0 stars the convective zones satisfy the Boussinesq approximations fairly well [*Spiegel*, 1960], though there are difficulties with boundary conditions and with radiative transfer. The observations of these stars indicate large velocity fluctuations and undetectable convective heat transfer, just as we would expect from our discussion. It would seem that for these stars, at least, our present considerations are relevant.

Another interesting feature of the low-Prandtl-number case is its implication for the Malkus theory. We have seen that as $\sigma \to 0$ the convective heat transfer vanishes, and fixing the boundary temperatures fixes the total heat transfer. Thus the concept of maximum heat transfer cannot be employed. Further, in the limit as $\sigma \rightarrow 0$ there is no boundary layering and the well-known $R^{1/3}$ dependence of heat transfer should not hold. Yet this does not seem deducible from the Malkus approach, which shows no Prandtl-number dependence on heat transport. It is interesting, however, that the first corrections (in σ) to the temperature gradient can be estimated from our present results, and these do indicate the incipience of a boundary layer.

We must nevertheless conclude that the Malkus theory in its present form does not apply to low-Prandtl-number convection. The reason for this lies in the important role of nonlinear terms in the low-Prandtl-number limit; these are the terms whose structure the Malkus theory does not treat.

Finally, we must comment on the calculation of the velocity spectrum. The transfer functions employed in the preceding section leave one dissatisfied, and it may be hoped that new ones, devised for the present purpose, will improve matters. But the difficulties go deeper than this. A principal source of inaccuracy arose in averaging over all directions in $\mathbf{k}$ space. This was necessary in order to employ the simpler approximations to the transfer function, but it caused a great loss of information. An alternative would be to use Kraichnan's direct-interaction approximation, which will permit us to preserve directional information. Unfortunately, the equations resulting from Kraichnan's approximation are very difficult to solve, even by numerical techniques. There seems to be no alternative at present but to attempt such a solution if the thermal turbulence problem is to be attacked directly.

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