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Intensification of the Earth's Magnetic Field by Turbulence in the Ionosphere[1]

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Abstract. If the magnetic diffusivity of a fluid in turbulent motion is greater than its kinematic viscosity, then, according to Batchelor, any magnetic field initially present must ultimately decay to zero. However, if a magnetic field is externally maintained, the turbulence will generate fluctuations that may be quite large if the magnetic Reynolds number is large compared with unity. The spectrum of these fluctuations increases up to a wave number k_c marking the threshold of conduction effects, and falls off rapidly beyond k_c. The net effect of the turbulence can be expressed in terms of an eddy conductivity equal to the molecular conductivity multiplied by the (—5/2)th power of the magnetic Reynolds number.

A situation which may well arise in ionospheric turbulence, but which has received little attention so far, is that described by the inequalities

$$1 \ll R_m \ll R \tag{1}$$

R and R_m are the turbulent Reynolds number and magnetic Reynolds number, defined in terms of the root-mean-square velocity u', the length-scale of the energy-containing eddies L, and the two diffusive constants characterizing the medium, ν the kinematic viscosity and λ the magnetic diffusivity:

$$R = u'L/\nu, \qquad R_m = u'L/\lambda \tag{2}$$

λ is inversely proportional to the electrical conductivity σ and the magnetic permeability μ of the medium:

$$\lambda = (4\pi\mu\sigma)^{-1} \tag{3}$$

According to the work of *Batchelor* [1950], the condition $R_m << R$ (that is, $\lambda >> \nu$) ensures that weak random magnetic fields will ultimately decay to zero. The arguments used by Batchelor were not universally accepted, but I shall assume their validity here without elaboration, and explore some of the consequences. Accepting, then, that random magnetic fields decay to zero, let us suppose that a large-scale weak magnetic field is applied to the system. (This is the case of interest in ionospheric studies, the applied field being then the earth's magnetic field.) It is as though a pulse of magnetic field were injected into the system in each time interval δt. Each pulse is distorted by the turbulence, being thoroughly randomized at the same time, and is then destroyed at the wave numbers at which conduction becomes important. The applied field must be weak if its dynamic effect is to be neglected.

The wave number marking the onset of viscous effects, k_v, is known to depend only on ν and ϵ, the rate of dissipation of energy per unit mass, through the equation

$$k_v = (\epsilon/\nu^3)^{1/4} \tag{4}$$

Since $\lambda >> \nu$, the wave number k_c marking the onset of conduction effects must be smaller than k_v, and, provided that it is at the same time large compared with L^{-1} (the wave number characteristic of the energy-containing eddies), it can only depend on ϵ and λ through the relationship

$$k_c = (\epsilon/\lambda^3)^{1/4} \tag{5}$$

The relations

$$L^{-1} \ll k_c \ll k_v \tag{6}$$

are equivalent to (1), if we accept the well-

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known semiempirical formula for ϵ,

$$\epsilon = u'^3/L \tag{7}$$

The picture that we contemplate is, then, the following. A steady source of magnetic energy is maintained at wave numbers of the order of L^{-1} or less. The associated field $\mathbf{H}_0(\mathbf{r})$ is distorted by the turbulence, each successive pulse of the field being driven by the turbulence into the wave-number range $k > k_c$, in which it is destroyed by the overriding effect of conduction. We seek to determine the spectral properties of the statistically steady magnetic field $\mathbf{H}(\mathbf{r}, t)$ (the sum of all the pulses) that is thus supported.

Now, in stating that the turbulence is statistically steady, we implicitly assume that there exists a source of vorticity at low wave numbers (i.e., that some large-scale destabilizing force generates vorticity). This vorticity is continuously transferred by the nonlinear interaction to the viscous sink at wave numbers $k > k_v$. In so far as viscous effects and conductive effects can be neglected, both vortex lines and magnetic field lines are convected in the fluid motion, the flux of either quantity through any material circuit being conserved. It is therefore fair to argue, in the manner adopted by Batchelor, that the steady statistical properties of the two fields (vorticity and magnetic) that are not influenced by the diffusive mechanisms must be identical. In particular, the spectra of vorticity and magnetic field must have the same wave-number dependence in the range defined by

$$L^{-1} \ll k \ll (\epsilon/\lambda^3)^{1/4} \tag{8}$$

Moreover, at wave numbers large compared with L^{-1}, if we accept Kolmogorov's picture, the velocity and vorticity spectrum tensors are isotropic, and so therefore is the magnetic field spectrum tensor. We can therefore define a scalar $\Gamma(k)$, the magnetic field spectrum, which increases as $k^{+1/3}$, like the vorticity spectrum, in the range (8). We anticipate that $\Gamma(k)$ will decrease rapidly in the range $k > k_c$ because of the severe conductive damping. For $k < L^{-1}$, the spectrum tensor will not in general be isotropic, but in this range $\Gamma(k)$ may be understood to represent the trace of the spectrum tensor averaged over the sphere of radius k in wave-number space. Then $\Gamma(k)\, dk$ is proportional to the contribution to the total magnetic energy $(1/8\pi)\, \overline{H^2}$ from the range $(k, k + dk)$:

$$\tfrac{1}{2}\overline{H^2} = \int_0^\infty \Gamma(k)\, dk \tag{9}$$

Clearly the maximum contribution comes from the region around $k = k_c$. Moreover, the rapid falloff that we anticipate for $k > k_c$ implies that, in any realization of the flow in physical space, the magnetic lines of force will be approximately straight with small fluctuations throughout any volume V_c of fluid of dimension smaller than k_c^{-1}. The spectrum $\Gamma(k)$ for $k >> k_c$ may be determined by a perturbation method based upon this premise. I need not give the details here; they closely resemble the work of *Golitsyn* [1960], who determined the form of the spectrum when it is wholly controlled by the conductive damping. The result is

$$\Gamma(k) = C\overline{H^2}\epsilon^{2/3}\lambda^{-2}k^{-11/3} \tag{10}$$

where C is a dimensionless constant of order unity.

Thus $\Gamma(k)$ rises as $k^{+1/3}$ up to k_c and falls as $k^{-11/3}$ beyond k_c. Since it must be continuous in order of magnitude near k_c, and since we must satisfy equation 9 so that C may be determined, $\Gamma(k)$ may finally be written in the form

$$\Gamma(k) = \begin{cases} 4/9\lambda\overline{H^2}\epsilon^{-1/3}k^{1/3} & L^{-1} \ll k \ll k_c \\ 4/9\lambda^{-2}\overline{H^2}\epsilon^{2/3}k^{-11/3} & k_c \ll k \ll k_v \end{cases} \tag{11}$$

It may fairly be supposed that $\Gamma(k)$ does not behave too erratically for $k < L^{-1}$ and that it falls off very rapidly for $k > k_v$.

The net effect of the turbulence on the field may be computed roughly as follows. In the absence of turbulence, only the large-scale field $\mathbf{H}_0(\mathbf{r})$ would be present with mean square value approximately

$$\overline{H_0^2} = \int_0^{L^{-1}} \Gamma(k)\, dk \doteqdot L^{-1}\Gamma(L^{-1}) \tag{12}$$

If we suppose that the spectral law 11(a) is valid, at least in order of magnitude, right up to $k = L^{-1}$, then, using (2) and (7), (12) becomes

$$\overline{H^2} = 9/4R_m\overline{H_0^2} \tag{13}$$

indicating the extent to which turbulence at large magnetic Reynolds number increases the mean square field intensity (provided always that $R_m < R$).

The extent to which the dissipation of energy by Ohmic heating is increased may also be readily calculated. Thus, if

$$D_0 = \lambda \int_0^{L^{-1}} k^2\Gamma(k)\, dk \doteqdot \lambda L^{-3}\Gamma(L^{-1}) \qquad (14)$$

is the dissipation in the absence of turbulence, and

$$D = \lambda \int_0^{\infty} k^2\Gamma(k)\, dk \doteqdot \lambda k_c^{\,3}\Gamma(k_c) \qquad (15)$$

is the increased dissipation under turbulent conditions, then

$$D = R_m^{\,5/2}\, D_0 \qquad (16)$$

This result may be interpreted in terms of an eddy conductivity σ_e, defined so that the total dissipation, by analogy with (14), is given by

$$D = (4\pi\mu\sigma_e)^{-1}L^{-3}\Gamma(L^{-1}) \qquad (17)$$

Then evidently σ_e can be expressed in the equivalent forms

$$\sigma_e = R_m^{\,-5/2}\sigma = (4\pi\mu u' L)^{-5/2}\sigma^{-3/2} \qquad (18)$$

indicating first the strong dependence of σ_e on R_m and second the anomalous effect that increase of σ, keeping other parameters constant, induces a decrease in σ_e, simply because the conduction cutoff k_c is raised, so that more thorough mixing of the magnetic field is possible.

It must be stressed that the above theory neglects the dynamic influence of the field on the fluid motion, a neglect that is justified only if the applied field is sufficiently weak. The salient features of the theory are, then, that the mean square value of the field is increased by a factor of order R_m and that its typical scale of variation is of order $(\lambda^3/\epsilon)^{1/4}$. The condition $R_m >> 1$ may only be satisfied in rare circumstances in the ionosphere where the turbulence is intense and the conductivity high. At the high altitudes at which this might be the case, the molecular mean free path is so large that the continuum approximations on which the theory is based must be viewed with caution. But the conclusion that intense fluctuations may at such high altitudes be superimposed on the earth's magnetic field is unlikely to be invalidated.

References

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