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Production of Longitudinal Vortices in the Boundary Layer along a Concave Wall[1]

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Abstract. Velocity measurements in the boundary layer along a concave wall reveal a spanwise variation having a definite wave number whether the boundary layer is laminar or turbulent. The variation is interpreted as produced by a system of vortices with axis in the streamwise direction. For the laminar boundary layer the vortices are identified with those predicted by the theory of laminar instability. The wave amplitude distribution and wave amplification rate are in good agreement with the theoretical prediction. For the turbulent boundary layer the concept of eddy viscosity is introduced to account for the observed phenomena by the theory of laminar instability. In both cases, however, the mechanism by which the wave number is determined remains an open question. Since the centrifugal force in the concavely curved boundary layer is analogous to the buoyant force in the thermally stratified boundary layer in giving rise to instability, some analogous phenomena may be expected in the boundary layer along a heated horizontal wall.

Introduction. In relation to geophysical applications, it was observed long ago that a system of longitudinal vortices (vortices with axis in the streamwise direction) is produced in the boundary layer along a heated horizontal wall [*Idrac*, 1921; *Terada*, 1928; *Terada and Tamano*, 1929; *Avsec and Luntz*, 1937]. Theoretical analysis by *Görtler* [1940] predicted that vortices of a similar nature would be produced also in the boundary layer along a concave wall as a consequence of the instability of laminar flow against small disturbances in the form of longitudinal vortices. It was only recently, however, that the relationship between the two phenomena was recognized by *Görtler* [1959], who demonstrated that the buoyant force in the thermally stratified layer is analogous to the centrifugal force in the concavely curved layer in giving rise to instability of laminar flow.

Experimental investigation has been undertaken on the boundary layer along a concave wall by the present author in collaboration with Y. Aihara and M. Iuchi. The investigation has three aspects. First, it aims at identifying the vortices observed in the laminar boundary layer with those predicted theoretically, for which no direct experimental verification has yet been provided except for their effect on transition to turbulent flow [*Liepmann*, 1945] or their traces as revealed by china-clay streaks on the wall [*Gregory and Walker*, 1956]. Second, it is considered worth while to investigate the growth of disturbances into turbulence in a boundary layer involving longitudinal vortices. Third, it may well be asked whether the vortices are also produced in the fully developed turbulent boundary layer. The present paper concerns the first and third problems; the second is to be discussed elsewhere.

Because of the analogy mentioned above, the results obtained for the concavely curved boundary layer will provide some qualitative information about the thermally stratified boundary layer, which is more closely related to geophysical problems. From the standpoint of experimental procedure, the concavely curved boundary layer appears preferable because of the ease of obtaining uniformity of flow conditions.

Experimental procedure. Measurements were made on two concave-wall models, one with a radius of curvature of 5 meters and the other of 10 meters. The 5-meter-radius model was mounted in a wind tunnel having a working section of 1 by 1 meter; the 10-meter-radius model, in a wind tunnel having a working section of 60

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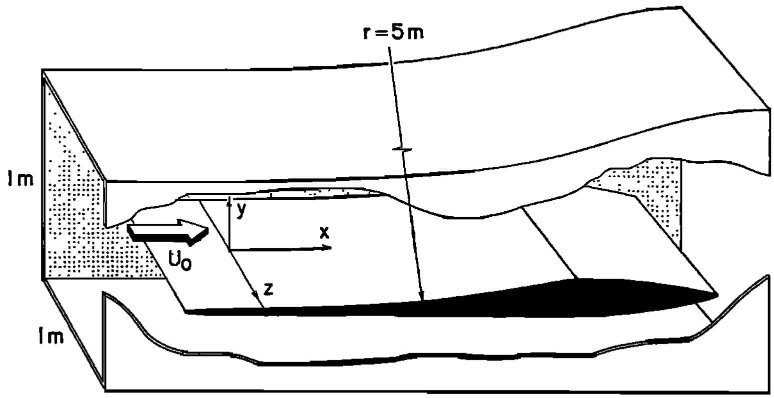

Fig. 1. Cutaway view of curved wall mounted in the wind tunnel.

by 60 cm. The 5-meter-radius model had a leading-edge portion of plane wall 40 cm long, but the 10-meter-radius model was curved immediately from the leading edge. By adjusting the wall opposite the test wall, it was possible to eliminate the streamwise gradient of free-stream velocity (velocity outside of the boundary layer) within the accuracy of measurements. Figure 1 shows a cutaway view of the 5-meter-radius model mounted in the wind tunnel. The reference axes are also shown, with the x axis along the wall in the streamwise direction (measured from the juncture of plane and curved walls), the y axis in the direction perpendicular to the wall, and the z axis in the spanwise direction.

Measurements were made at the free-stream velocity U_0 of 3 to 20 m/s for the 5-meter-radius model, and 11 to 16 m/s for the 10-meter-radius model. The free-stream turbulence level was 0.3 per cent and 0.05 per cent, respectively. Since the boundary layer remained mostly laminar for the free-stream velocities used, it was made artificially turbulent, if necessary, by placing a piano wire on the test wall near the leading edge. The diameter of the wire was chosen slightly larger than the minimum value that would make the flow turbulent immediately behind. Survey of velocity profiles a short distance downstream from the wire indicated that the flow was fully turbulent and free from any spanwise variation. It should in particular be noted that no periodic variation in spanwise direction could be detected.

Velocity profiles were determined by means of a total-pressure tube and a static-pressure tube.

Vortices in laminar boundary layer. Figure 2 shows a typical result for the station $x = 90$ cm of the streamwise component of velocity U obtained by traversing a total-pressure tube in the z direction at a fixed height y from the concave wall with a radius of curvature r of 10 meters. The periodic variation of velocity is considered to be produced by the system of alternating longitudinal vortices superposed on the Blasius flow along the wall. The same result, together with results obtained for other stations, is presented in Figure 3 in the form of lines of constant relative velocity U/U_0. It is seen from these data that the effect of vortices, most no-

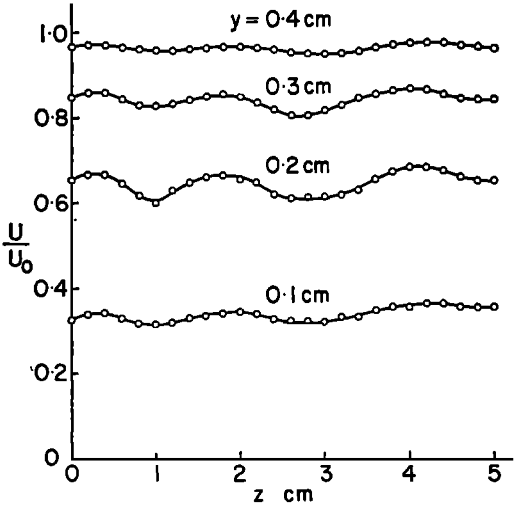

Fig. 2. Velocity distribution in transverse direction at a fixed height from the wall. $x = 90$ cm, $r = 10$ meters, $U_0 = 11$ m/s. Laminar boundary layer.

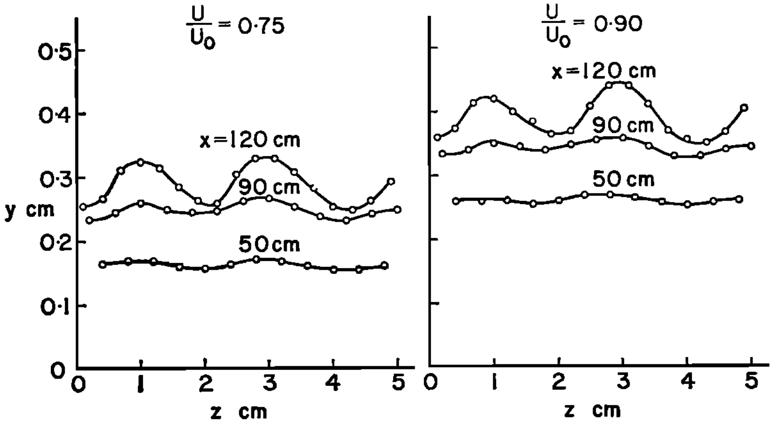

Fig. 3. Lines of constant velocity at three stations, $x = 50$, 90, and 120 cm. $r = 10$ meters, $U_0 = 11$ m/s. Laminar boundary layer.

ticeable somewhere in the center of the boundary-layer thickness, increases exponentially in the downstream direction.

The wavelength of spanwise variation is about 2.1 cm, and the number of waves a contained within a distance of 2π cm is $2\pi/2.1 = 3.0$. The corresponding value for the wall of $r = 5$ meters is $2\pi/3.4 = 1.8$. The wave number is independent of x, and nearly independent of U_0.

Figure 4 presents the Görtler parameter $G \equiv R_\theta(\theta/r)^{1/2}$ as a function of the nondimensional wave number $a\theta$, where θ is the momentum thickness of the boundary layer (average value in the spanwise direction), and $R_\theta \equiv U_0\theta/\nu$ is the Reynolds number based on free-stream velocity and momentum thickness. Comparison is made with the prediction by small-disturbance theory, which was originally due to *Görtler* [1940]

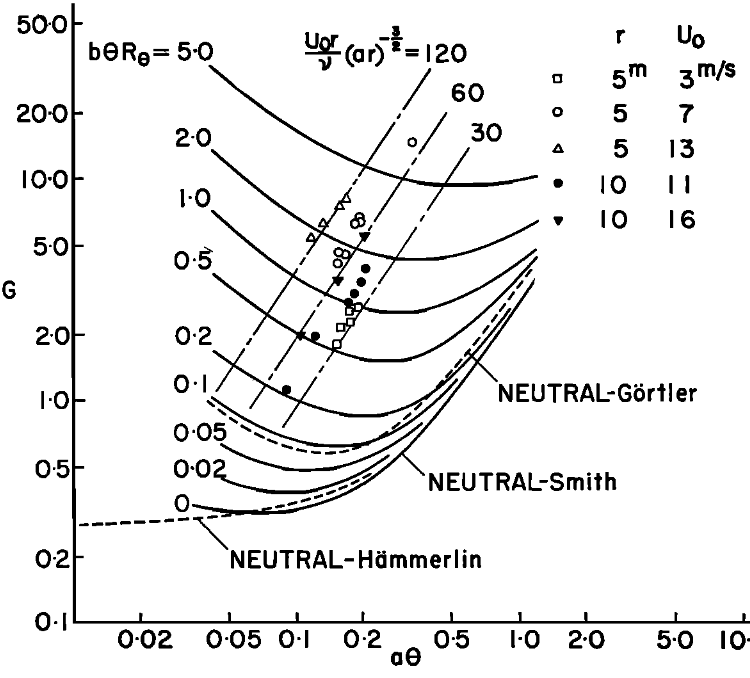

Fig. 4. Comparison of wave number of vortices observed on the concave wall with theoretical prediction. $r = 5$ and 10 meters. Laminar boundary layer.

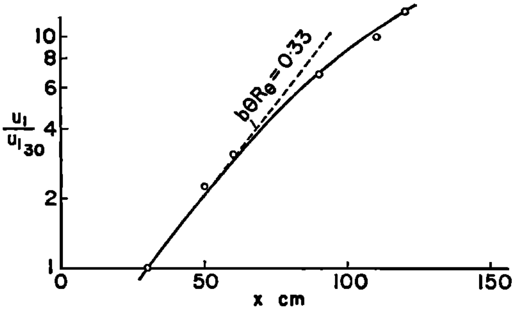

Fig. 5. Determination of wave amplification rate by plotting maximum wave amplitude u_1 against x in semilogarithmic scale. $r = 10$ meters, $U_0 = 11$ m/s. Laminar boundary layer.

and subsequently modified by *Hämmerlin* [1955] and *Smith* [1955]. For small values of $a\theta$, Hämmerlin's result is more accurate than Görtler's. In both theories the basic flow and disturbance are invariant in the x direction, the disturbance being assumed in the form $F(y)e^{Bt} \cos az$, where t is the time. The disturbance is amplified or damped in the course of time according as B is positive or negative; it is neutral for $B = 0$. In Smith's theory, the basic flow is invariant in the x direction, but the disturbance is assumed in the form $F(y)e^{bx} \cos az$. The disturbance is amplified or damped in the course of downstream development according as b is positive or negative; it is neutral for $b = 0$. The assumptions made in Smith's theory appear to be closer to the conditions of the present experiment.

It is seen from Figure 4 that all the experimental points are located in the region where the disturbance is to be amplified according to theory. This affords evidence to justify the observed result. In order to obtain further evidence, the wave amplitude u/U_0 is read off from such a diagram as Figure 2. By determining the maximum value u_1/U_0 and plotting it against x in a semilogarithmic scale (Fig. 5), we obtain from the tangent the amplification rate b as $b\theta R_\theta = 0.33$, for which Smith's theory gives a value 0.38. In Figure 6 the distribution of the relative wave amplitude u/u_1 is compared with Görtler's theoretical distribution $F(y)$, since in Smith's paper no calculation has been worked out for the amplitude distribution relevant to the observed result. The agreement between observed and theoretical results appears satisfactory.

Thus, the observed spanwise variation of velocity is identified with that produced by the system of vortices predicted theoretically. However, the theory predicts nothing about the wave number that will actually come out for a given radius of curvature and a given free-stream velocity. As was mentioned above, the wave number is 1.8 and 3.0 per 2π cm for the radius of curvature of 5 and 10 meters, respectively, almost independently of the free-stream velocity. Additional measurements were made on the 10-meter-radius model by reducing the spanwise dimension of the wind tunnel by inserting false walls, but no appreciable change was found in wave number. Consequently, the mechanism by which the wave number is determined still remains an open question.

Vortices in turbulent boundary layer. As far as the author is aware, neither theoretical prediction nor experimental evidence has been advanced indicating whether the longitudinal vortices are produced also in the turbulent boundary layer along a concave wall. The Reynolds number above which a disturbance in the laminar boundary layer will be amplified is relatively low (Fig. 4), and, moreover, some sort

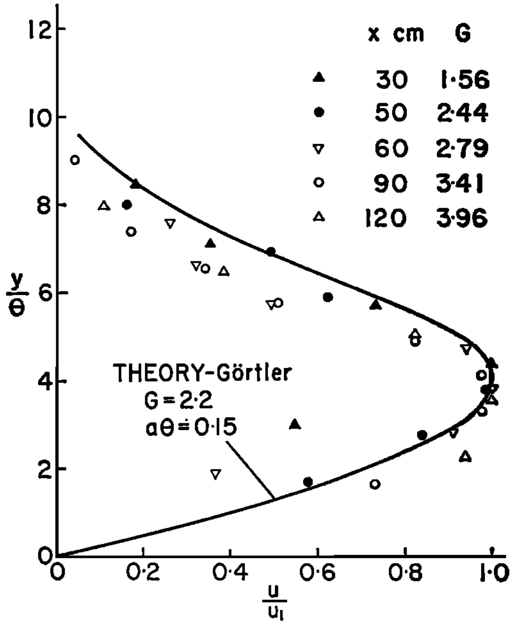

Fig. 6. Comparison of wave amplitude distribution of vortices observed on the concave wall with theoretical prediction. $r = 10$ meters, $U_0 = 11$ m/s. Laminar boundary layer.

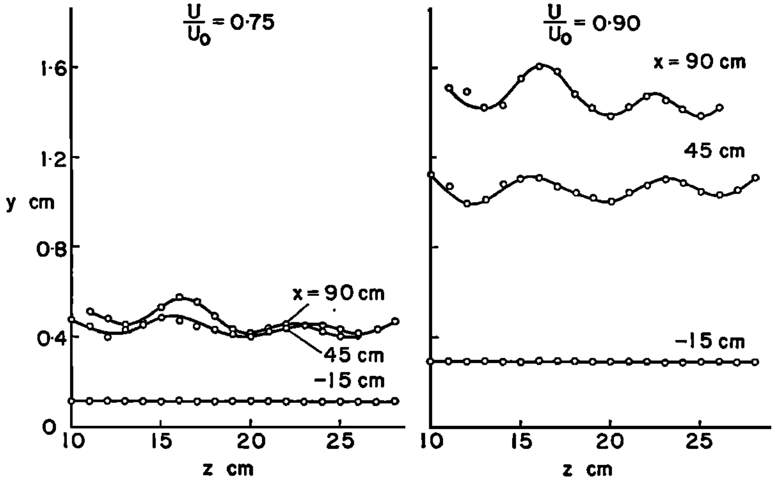

Fig. 7. Lines of constant velocity at three stations, $x = -15$, 45, and 90 cm. $r = 5$ meters, $U_0 = 20$ m/s. Turbulent boundary layer.

of turbulent flow behaves like a laminar flow of low Reynolds number, which might be obtained by replacing the kinematic viscosity by the turbulent viscosity, or eddy viscosity. This reasoning has led the author to make a simple calculation to the following effect.

It is known that the stability characteristics of a concavely curved laminar boundary layer are rather insensitive to the change in velocity profile in the y direction, if the momentum thickness is used as the reference length. It can therefore be assumed that the stability diagram as shown in Figure 4 applies also to a concavely curved turbulent boundary layer only if the kinematic viscosity ν is replaced by the eddy viscosity ν_T. The velocity profile of the outer 80 to 90 per cent of the turbulent boundary layer is accurately predicted by the simple assumption of a constant eddy viscosity, for which *Clauser* [1956] suggests a value $\nu_T = 0.018\ U_0\delta^*$, where δ^* is the displacement thickness, irrespective of the Reynolds number, pressure

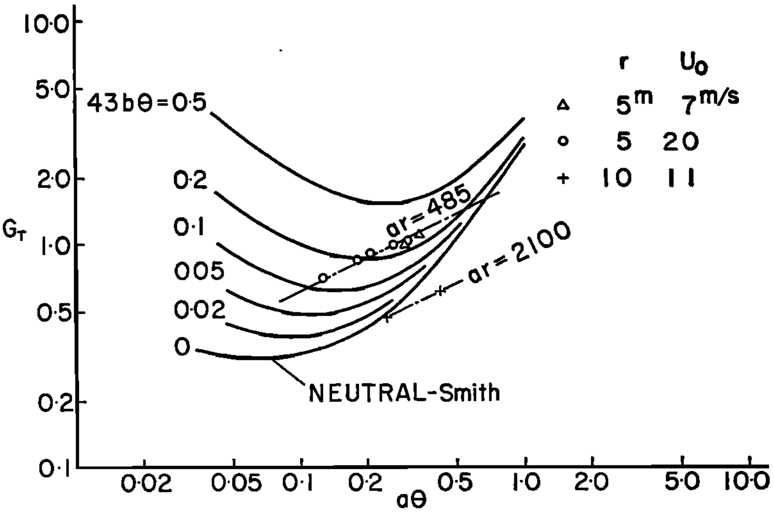

Fig. 8. Comparison of wave number of vortices observed on the concave wall with theoretical prediction. $r = 5$ and 10 meters. Turbulent boundary layer.

gradient, or roughness of the wall. With this value for ν_T, together with the additional assumption that $\delta^* = 1.3\ \theta$, the Reynolds number involved in the Görtler parameter is now to be replaced by $1/(0.018 \times 1.3) = 43$, so that the Görtler parameter for turbulent boundary layer is given by $G_T = 43(\theta/r)^{1/2}$. For obtaining a typical value of $G_T = 0.5$, for example, θ is to be 0.1 cm for $r = 5$ meters. This is within the range of the present experiment.

Experimental results on turbulent boundary layer are presented in the same way as for the laminar boundary layer. Figure 7 illustrates the lines of constant velocity for the three stations, $x = -15$, 45, and 90 cm, along the concave wall of $r = 5$ meters. The lines are nearly flat at the foremost station $x = -15$ cm located on the plane wall, but exhibit a distinct waviness at the downstream stations on the curved wall. The wavelength is about 6.5 cm, so that the wave number per 2π cm is $a = 2\pi/6.5 = 1.0$. This wave-number value appears to be only slightly affected by the change in free-stream velocity. For the concave wall of $r = 10$ meters, the wave amplitude is so small that the wave number is only roughly estimated at 2 per 2π cm.

The results on wave number are presented in Figure 8 by plotting the turbulent Görtler parameter G_T against the nondimensional wave number $a\theta$. Smith's result for laminar flow is entered for reference. The experimental points for $r = 5$ meters are located in the amplification region; those for $r = 10$ meters are close to the neutral curve.

Since the wave number does not vary in the x direction, the sequence of events in downstream developments is indicated by a straight line of slope 1/2 in the logarithmic plot of G_T against $a\theta$. This is in marked contrast to the case of laminar flow, in which the sequence of events is given by a straight line of slope 3/2 (chain lines in Fig. 4). This difference seems in favor of the wave amplification in the laminar boundary layer.

Conclusion. Velocity measurements in the boundary layer along a concave wall reveal a spanwise variation having a definite wave number whether the boundary layer is laminar or turbulent. The variation is interpreted as produced by a system of vortices with axis in the streamwise direction. For the laminar boundary layer the vortices are identified with those predicted by the theory of laminar instability. The wave amplitude distribution and wave amplification rate are in good agreement with theoretical prediction. For the turbulent boundary layer the concept of eddy viscosity is introduced to account for the observed phenomena by the theory of laminar instability. For both, however, the mechanism by which the wave number is determined remains an open question.

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