Some Mathematical Models Generalizing the Model of Homogeneous and Isotropic Turbulence[1]

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Abstract. Homogeneous and isotropic turbulence is an example of a system of random fields invariant with respect to a group of motions. Along with homogeneous and isotropic fields, locally homogeneous and locally isotropic ones play an important role in turbulence theory; such local fields may also have an accurate mathematical definition. The random fields invariant with respect to groups of transformations different from a group of Euclidean motions can also be considered; the 'spectral representation' of such a field and of a corresponding correlation function often has an unusual form, although its sense remains the same. The algebraic theory of group representations gives the general method of obtaining the spectral representation for the fields. The examples of homogeneous random fields on a sphere and fields in a semiplane invariant with respect to all similarity transformations present interesting examples of random fields invariant with respect to 'motions' of special type which might be of some importance for turbulence theory.

1. It is well known how important for the development of the statistical theory of turbulence was the idea of *Taylor* [1953], who suggested the investigation of the simplest model of homogeneous and isotropic turbulence. So far almost all the results obtained in the statistical theory refer either to the particular model or to some of its simple generalizations. In the present paper we shall try to find out why homogeneous and isotropic turbulence is so convenient for theoretical study and shall point out some general mathematical considerations that make it possible to investigate a number of other turbulent flows satisfying certain statistical symmetry conditions.

The most important advantage of homogeneous and isotropic turbulence is obviously that the presence of strict symmetry conditions sharply decreases both the number of different moments (multiple correlations) characterizing the turbulence and the number of variables upon which the correlations depend. For example, the general turbulent flow of a compressible fluid has six different double velocity correlations, depending upon seven variables, whereas in the homogeneous and isotropic case we shall deal with only two different double velocity correlations depending upon the two variables r and t. It is clear that the stricter symmetry conditions imposed on turbulent flow involve a simpler form for all its statistical characteristics and thus make the development of a comprehensive statistical theory of a corresponding turbulence more probable. It is primarily due to this fact that the full statistical description of arbitrary turbulent flows still remains a rather hopeless problem, and all the concrete results of statistical theory available refer to particular classes of flows, satisfying rather strict symmetry conditions. The case of homogeneous and isotropic turbulence has the further important advantage over the general case that effective description of all the double correlations admitted for a homogeneous and isotropic turbulence is possible, and this is closely related to the existence of the spectral representation of homogeneous and isotropic random fields. We shall discuss this last advantage and some of its generalizations.

2. The term 'random field in the n-dimensional space R_n' will be used for the set of random variables $u(\mathbf{x})$ corresponding to all the points $\mathbf{x} = (x_1, \cdots, x_n)$ of R_n and having definite probability distributions for field values in any final sets $\mathbf{x}_1, \cdots, \mathbf{x}_m$ of points. It is clear that

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for the theory of turbulence only the cases of two-, three-, and four-dimensional space R_n are of interest. For simplicity we shall consider only random fields with zero mean values $\overline{u(\mathbf{x})} = 0$ (the bar above will always denote mathematical expectation); the last condition will be fulfilled, for example, for the field of fluctuations of any hydrodynamical quantity of turbulent flow. The most important statistical characteristic of the random field $u(\mathbf{x})$ is its correlation function, or more precisely double correlation function,

$$B_{uu}(\mathbf{x}', \mathbf{x}'') = \overline{u(\mathbf{x}')u(\mathbf{x}'')} \qquad (1)$$

This function depends on $2n$ variables $x_1', \cdots, x_n', x_1'', \cdots, x_n''$. However, it is not an arbitrary function of $2n$ variables but positive definite, that is, such that, for any integer m, any m points $\mathbf{x}_1, \cdots, \mathbf{x}_m$ from R_n and any m real numbers $a_1, \cdots, a_m$, the inequality

$$\sum_{j,l=1}^{m} B_{uu}(\mathbf{x}_j, \mathbf{x}_l)a_j a_l \geq 0 \qquad (2)$$

holds. Unfortunately, condition 2 is not effective: in the general case for the given function $B_{uu}(\mathbf{x}', \mathbf{x}'')$ it is difficult to verify whether condition 2 holds or not. Therefore for general turbulent flow it is usually difficult to say whether the correlation functions computed theoretically from some hypothesis which seems quite reasonable satisfy this necessary condition.

Now let us suppose that the field $u(\mathbf{x})$ is homogeneous, that is, such that the function $B_{uu}(\mathbf{x}', \mathbf{x}'')$ depends only on the difference $\mathbf{r} = \mathbf{x}'' - \mathbf{x}'$. Then according to the Bochner-Khinchine theorem in the case of the continuous spectrum (which is the only one required for the turbulence theory) the function $B_{uu}(\mathbf{r})$ will be positive definite if and only if it can be represented in the form

$$B_{uu}(\mathbf{r}) = \int e^{i\mathbf{k}\mathbf{r}} F_{uu}(\mathbf{k})\, d\mathbf{k} \qquad (3)$$

where $F_{uu}(\mathbf{k})$ is a nonnegative function, that is, when the Fourier transform of the function $B_{uu}(\mathbf{r})$ is everywhere nonnegative. In the case of the set $\mathbf{u}(\mathbf{x}) = \{u_1(\mathbf{x}), \cdots, u_m(\mathbf{x})\}$ of several homogeneous and homogeneously connected random fields the similar requirement to the correlation matrix

$$B_{ij}(\mathbf{r}) = \overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})} \qquad (4)$$

has the following form: the functions $B_{ij}(\mathbf{r})$ must be represented as the integrals

$$B_{ij}(\mathbf{r}) = \int e^{i\mathbf{k}\mathbf{r}} F_{ij}(\mathbf{k})\, d\mathbf{k} \qquad (5)$$

where the matrix $||F_{ij}(\mathbf{k})||$ at any $\mathbf{k}$ is nonnegative definite.

From the possibility of the spectral representations 3 and 5 of the correlation functions $B_{uu}(\mathbf{r})$ and the correlation matrices $B_{ij}(\mathbf{r})$, it follows that the homogeneous fields $u(\mathbf{x})$ and $u_j(\mathbf{x})$ can be represented in the form of Fourier-Stieltjes integrals

$$u(\mathbf{x}) = \int e^{i\mathbf{k}\mathbf{x}} Z_u(d\mathbf{k}) \quad u_j(\mathbf{x}) = \int e^{i\mathbf{k}\mathbf{x}} Z_j(d\mathbf{k}) \qquad (6)$$

Here $Z_u(d\mathbf{k})$ is the random complex additive function of n-dimension interval $d\mathbf{k}$ satisfying the relation

$$\overline{Z_u(d\mathbf{k}')Z_u^*(d\mathbf{k}'')} = \delta(\mathbf{k}' - \mathbf{k}'')F_{uu}(\mathbf{k}')\, d\mathbf{k}'\, d\mathbf{k}'' \qquad (7)$$

where $\delta(\mathbf{k})$ is the Dirac function, the asterisk is the sign of a complex conjugation, and $Z_j(d\mathbf{k})$, $j = 1, \cdots, m$ is the system of the similar functions for which condition 7 is replaced by

$$\overline{Z_j(d\mathbf{k}')Z_l^*(d\mathbf{k}'')} = \delta(\mathbf{k}' - \mathbf{k}'')F_{jl}(\mathbf{k}')\, d\mathbf{k}'\, d\mathbf{k}'' \qquad (8)$$

The spectral representation (6) of the fields themselves together with relations 7 and 8 show that the spectral functions $F_{uu}(\mathbf{k})$ and $F_{jj}(\mathbf{k})$ have a clear physical meaning of the distribution of fluctuation energy over the spectrum of wave vectors; the spectrum functions $F_{jl}(\mathbf{k})$ have a similar meaning. The possibility of using the ordinary analytical tool of Fourier transforms, arising from (6), for the study of homogeneous turbulence facilitates essentially the development of a mathematical theory of homogeneous turbulence; for this reason the requirement of homogeneity is of greatest importance in the statistical theory of turbulence; see for example *Batchelor* [1953].

3. Let us now suppose that the fields $u(\mathbf{x})$ and $\mathbf{u}_j(\mathbf{x})$ are not only homogeneous but also isotropic. In the case of a scalar field $u(\mathbf{x})$, such, for example, as the field of pressure or temperature fluctuations, the function $B_{uu}(\mathbf{r})$ must depend only on $r = |\mathbf{r}|$ and $F_{uu}(\mathbf{k})$ must depend only on $k = |\mathbf{k}|$. After integrating in (3) on all

the angular variables we find that the class of correlation functions $B_{uu}(r)$ of a homogeneous and isotropic scalar field in R_n coincides with the class of function represented in the form

$$B_{uu}(r) = \int_0^\infty \frac{J_{(n-2)/2}(kr)}{(kr)^{(n-2)/2}} \Phi(k)\, dk \qquad (9)$$

where $J(n - 2)/2$ is the Bessel function of the order $(n - 2)/2$, and $\Phi(k)$ is any nonnegative function; see *Schoenberg* [1938]. The function $\Phi(k)$, which differs only by numerical multiples from $k^{n-1} F_{uu}(k)$, can be determined from $B_{uu}(r)$ with the help of the following conversion formula corresponding to (9):

$$\Phi(k) = \int_0^\infty (kr)^{n/2} J_{(n-2)/2}(kr) B_{uu}(r)\, dr \qquad (10)$$

Equations 9 and 10 with $n = 2$ or $n = 3$ are of interest for the statistical theory of turbulence; still more important, however, are the similar equations for vector homogeneous and isotropic random fields $\mathbf{u}(\mathbf{x}) = \{u_1(\mathbf{x}), \cdots, u_n(\mathbf{x})\}$ in R_n, in view of the particular importance of the investigation of velocity fluctuations in isotropic turbulence. For such vector homogeneous and isotropic fields the matrix $B_{ij}(\mathbf{r})$ can be expressed by means of the scalar longitudinal and transversal correlation functions $B_{ll}(r)$ and $B_{nn}(r)$ by the familiar equation

$$B_{ij}(\mathbf{r}) = \frac{B_{ll}(r) - B_{nn}(r)}{r^2} r_i r_j + B_{nn}(r)\, \delta_{ij} \quad (11)$$

found by *von Kármán and Howarth* [1938]. The general form of the functions $B_{ll}(r)$ and $B_{nn}(r)$ in (11) is given by the equations

$$B_{ll}(r) = \int_0^\infty \left[\frac{J_{n/2}(kr)}{(kr)^{n/2}} - \frac{J_{(n+2)/2}(kr)}{(kr)^{(n-2)/2}}\right]\Phi_{ll}(k)\, dk + (n-1)\int_0^\infty \frac{J_{n/2}(kr)}{(kr)^{n/2}} \Phi_{nn}(k)\, dk \qquad (12)$$

$$B_{nn}(r) = \int_0^\infty \frac{J_{n/2}(kr)}{(kr)^{n/2}} \Phi_{ll}(k)\, dk + \int_0^\infty \left[\frac{J_{(n-2)/2}(kr)}{(kr)^{(n-2)/2}} - \frac{J_{n/2}(kr)}{(kr)^{n/2}}\right]\Phi_{nn}(k)\, dk \qquad (13)$$

where $\Phi_{ll}(k)$ and $\Phi_{nn}(k)$ are two arbitrary nonnegative functions, which can easily be determined from $B_{ll}(r)$ and $B_{nn}(r)$ with the help of the special conversion equations similar to (10) (see *Yaglom* [1957]; for the case $n = 3$, particularly important for the theory of turbulence, corresponding equations have been given by *Yaglom* [1948] and *Moyal* [1952]). Thus, to verify whether the two given functions $B_{ll}(r)$ and $B_{nn}(r)$ can be the longitudinal and transversal correlation functions of the homogeneous and isotropic vector field in R_n it is only necessary to evaluate the corresponding functions $\Phi_{ll}(k)$ and $\Phi_{nn}(k)$ and to see whether both of them would be nonnegative or not.

When the given vector field is incompressible, that is, solenoidal, formulas 12, 13, and the corresponding conversion formulas are especially simple, since then $\Phi_{ll}(k) \equiv 0$; see *Batchelor* [1953].

As the homogeneous and isotropic random fields $u(\mathbf{x})$ and $u_i(\mathbf{x})$ are naturally homogeneous they can be represented in the spectral form (6). Only this form, which does not take into account the condition of isotropy, is generally used in works on turbulence. If, however, we do take the isotropy of the fields in formulas 6 into consideration they will assume a more special form, which is not very popular but may be useful in some turbulence problems. So the isotropic scalar field $u(\mathbf{x})$ on the two-dimensional plane R_2 and in three-dimensional space R_3 in the polar or, correspondingly, spherical coordinates will be written

$$u(\mathbf{x}) = u(r, \varphi) = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^\infty J_m(kr)\, dZ_m(k) \qquad (14)$$

and

$$u(\mathbf{x}) = u(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \gamma_{lm} Y_l^m(\theta, \varphi) \cdot \int_0^\infty \frac{J_{l+1/2}(kr)}{(kr)^{1/2}}\, dZ_{lm}(k) \qquad (15)$$

where $Y_l^m(\theta, \varphi)$ are spherical functions

$$\gamma_{lm} = \left[\frac{\sqrt{2\pi}\,(l + \frac{1}{2})}{2^{2l}} \frac{(l+m)!\,(l-m)!}{(l!)^2}\right]^{1/2}$$

and $Z_m(k)$, $Z_{lm}(k)$ are random functions with noncorrelated increments satisfying the equations

$$\overline{dZ_m(k')\, dZ_n^*(k'')} = \delta_{mn}\, \delta(k' - k'')\Phi(k')\, dk'\, dk'' \qquad (16)$$

and

$$\overline{dZ_{lm}(k')\, dZ^*_{jn}(k'')} = \delta_{lj}\, \delta_{mn}\, \delta(k' - k'')\Phi(k')\, dk'\, dk'' \qquad (17)$$

(see *Yaglom* [1961]). Similar spectral representations may also be written for vector homogeneous and isotropic fields; in this case instead of ordinary spherical functions so-called vector spherical harmonies must be used; see section 6.

4. In the statistical theory of turbulence, homogeneous and isotropic turbulence is studied much more thoroughly than other types. For such turbulence the class of all possible correlation functions of any scalar hydrodynamical field coincides with the class of functions of the form (9), where $n = 3$ and $\Phi(k) \geq 0$, and the class of possible longitudinal and transverse correlation functions of the velocity field coincides with the class of functions of the form (12) and correspondingly (13), where $n = 3$ and $\Phi_{ll}(k) \geq 0$, $\Phi_{nn}(k) \geq 0$ for compressible fluids and $\Phi_{ll}(k) = 0$, $\Phi_{nn}(k) \geq 0$ for incompressible fluids. Besides this, there is a series of works studying axisymmetric turbulence, that is, homogeneous turbulence invariant to all rotations on the axis parallel to a given direction Oz. The correlation functions $B_{uu}(\mathbf{r}) = B_{uu}(x, y, z)$ of such axisymmetric turbulence will evidently depend on two arguments z and $\rho = \sqrt{x^2 + y^2}$. The functions $B_{uu}(\rho, z)$ as the functions of the variable ρ will have the form (9) with $n = 2$, and as the functions of z they will be expanded into ordinary Fourier integrals. From this it follows that the general expression for the correlation function $B_{uu}(\rho, z)$ can be given by the formula

$$B_{uu}(\rho, z) = \int_{-\infty}^{\infty} \int_0^{\infty} e^{ikz} J_0(\kappa\rho)\Phi(\kappa, k)\, d\kappa\, dk \qquad (18)$$

where $\Phi(\kappa, k)$ is nonnegative function of two variables. Similarly the correlation matrix $B_{ij}(\mathbf{r})$ of the velocity field of axisymmetric turbulence can be expressed with the four scalar functions $B_{zz}(\rho, z)$, $B_{ll}(\rho, z)$, $B_{nn}(\rho, z)$, and $B_{lz}(\rho, z)$ of two variables ρ and z, where the function $B_{zz}(\rho, z)$ will have the form (18), the function $B_{ll}(\rho, z)$ will have the form

$$B_{ll}(\rho, z) = \int_{-\infty}^{\infty} \int_0^{\infty} e^{ikz} \left[\frac{\partial J_1(\kappa\rho)}{\partial(\kappa\rho)}\, \Phi_1(\kappa, k) + \frac{J_1(\kappa\rho)}{\kappa\rho}\, \Phi_2(\kappa, k) \right] dk\, d\kappa \qquad (19)$$

with nonnegative $\Phi_1(\kappa, k)$ and $\Phi_2(\kappa, k)$, and the functions $B_{nn}(\rho, z)$ and $B_{lz}(\rho, z)$ will have analogous forms.

For meteorological and oceanographical applications the more general case of turbulence in the semispace over the given 'wall' is of much greater interest. Such turbulence is invariant to all rotations and translations in the plane Oxy, but it is not invariant to translations along the axis Oz. In this case the expansion into the Fourier integral (3) and the more general forms (9) or (12) and (13) with $n = 2$ is possible only with respect to the variables x, y, and ρ. Moreover, the correlation functions $B_{uu}(\rho, z', z'')$, $B_{ll}(\rho, z', z'')$, etc., will depend on two heights z' and z'', and in respect to this last dependence the condition of positive definiteness will be ineffective. In section 7, however, we shall see that in some meteorological problems the corresponding turbulence in semispace will satisfy additional symmetry conditions of a special kind, permitting us to obtain the explicit formula for the most general correlation functions.

5. It is known that the general formulas 3 and 6 for the theory of the homogeneous random field in n-dimensional Euclidean space may be generalized for the case of homogeneous random fields $u(g)$ over an arbitrary locally compact commutative group G; see *Weil* [1940], *Raikow* [1945], and *Kampé de Fériet* [1947]. For such fields the exponents $e^{ik\mathbf{r}}$ and $e^{i\mathbf{kx}}$ in formulas 3 and 6 need only be replaced by the characters $\chi_k(g)$ of group G. However, in the statistical theory of turbulence we usually deal not with the homogeneous fields over the group but with the homogeneous fields $u(\mathbf{x})$ over some homogeneous space $\mathfrak{R}$ with the given group G (usually noncommutative) of motions. For such homogeneous fields $u(\mathbf{x})$ the correlation function $B_{uu}(\mathbf{x}', \mathbf{x}'')$ satisfies the equation

$$B_{uu}(\mathbf{x}', \mathbf{x}'') = B_{uu}(g\mathbf{x}', g\mathbf{x}'') \qquad (20)$$

The works of *Krein* [1950], *Gelfand* [1950], and *Yaglom* [1948; 1961] are devoted to the study of the general form of positive definite functions $B_{uu}(\mathbf{x}', \mathbf{x}'')$ over different spaces $\mathfrak{R}$ satisfying (20); in *Yaglom* [1948; 1961] the question about the analogies of the spectral representation (6) for the fields themselves is also considered. In the concrete applications to the problems of turbulence theory the space $\mathfrak{R}$ is usually symmetrical Riemannian space; according to the work of *Gelfand* [1950] the general form of the corresponding correlation functions $B_{uu}(\mathbf{x}', \mathbf{x}'')$ of scalar homogeneous fields $u(\mathbf{x})$ in the case of continuous spectrum can be written

$$B_{uu}(\mathbf{x}', \mathbf{x}'') = \int_K H^{(k)}(\mathbf{x}', \mathbf{x}'')\Phi(k)\, dk \qquad (21)$$

In this formula $H^{(k)}(\mathbf{x}', \mathbf{x}'')$ are all possible zonal spherical functions over $\mathfrak{R}$, defined in the works of *Godement* [1952], *Berezin and Gelfand* [1956], and others, k is a parameter enumerating different zonal spherical functions, K is the set of all k, and $\Phi(k)$ is a nonnegative function on K. The homogeneous random field $u(\mathbf{x})$ can be written

$$u(\mathbf{x}) = \int_K \sum_l H_l^{(k)}(\mathbf{x}) Z_l(dk) \qquad (22)$$

where the functions $H_l^{(k)}(\mathbf{x})$ with different l are all possible spherical functions corresponding to the given zonal spherical function $H^{(k)}(\mathbf{x}, \mathbf{x}_0) = H_0^{(k)}(\mathbf{x})$, and $Z_l(dk)$ are random additive functions of the set dk satisfying the condition

$$\overline{Z_l(dk')Z_l^*(dk'')} = \delta_{l,}\, \delta(k' - k'')\Phi(k')\, dk'\, dk'' \qquad (23)$$

(see *Yaglom* [1961]).

Where the space $\mathfrak{R}$ is ordinary Euclidean space R_n formulas 22 and 23 change into formulas 10 and 14–15. If $\mathfrak{R}$ is a non-Euclidean Lobachevski plane, formulas 22 and 23 give

$$B_{uu}(r) = \int_0^\infty P_{-1/2+\sqrt{1/4-k}}(\cosh r)\Phi(k)\, dk \qquad (24)$$

$$u(r, \varphi) = \sum_{l=-\infty}^{\infty} \gamma_l e^{-il\varphi} \cdot \int_0^\infty P^l_{-1/2+\sqrt{1/4-k}}(\cosh r)\, dZ_l(r) \qquad (25)$$

where r is the non-Euclidean distance, (r, φ) are non-Euclidean polar coordinates, $P_\mu{}^\nu$ are the special solutions of the Legendre differential equation, $P_\mu = P_\mu{}^0$, and γ_l are normalizing constants expressed by means of Γ function [*Krein*, 1950; *Yaglom*, 1961]. And if $\mathfrak{R}$ is a sphere S of Euclidean space R_3, formulas 22 and 23 give the following expressions for the correlation function $B_{uu}(\theta)$ of an arbitrary homogeneous field over S and for the field $u(\theta, \varphi)$ itself:

$$B_{uu}(\theta) = \sum_{l=0}^{\infty} \Phi_l P_l(\cos\theta) \qquad (26)$$

$$u(\theta, \varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Z_{lm} Y_l^m(\theta, \varphi) \qquad (27)$$

Here Φ_l are nonnegative constants and Z_{lm} are mutually noncorrelated random variables with covariances depending only on l. The results (26) and (27) obtained earlier by *Schoenberg* [1942] and *Obuchov* [1947] may have some applications to geophysics in connection with investigations of the statistical macrostructure of the meteorological fields all over the world (the influence of zonal climatic belts can then be taken into account as a small additive perturbation).

6. Formulas similar to (22) and (23) can also be obtained for vector homogeneous random fields on the wide class of the spaces $\mathfrak{R}$ with the transitive group G of motions. Then instead of ordinary spherical functions we must use some more general functions of the same kind; see *Yaglom* [1961]. Here we shall briefly consider only the problem about the general form of vector homogeneous fields $\mathbf{u}(\theta, \varphi)$ on the sphere S; the solution of this problem may be applied to the investigation of the very large-scale characteristics of the upper winds over the world. The vertical component of the vector $\mathbf{u}$ evidently forms the scalar field over S studied in section 5, so that we must investigate only the two-dimensional field $\mathbf{u}(\theta, \varphi) = \{u_\theta(\theta, \varphi), u_\varphi(\theta, \varphi)\}$, where u_θ and u_φ are components of the vector $\mathbf{u}$ along the meridian and parallel. The functions more general than the usual spherical functions mentioned above in this case will be the well-known spherical vector harmonics (see *Morse and Feshbach* [1953]; *Gelfand, Minlos, and Shapiro* [1958]). According to the notation of *Gelfand, Minlos, and Shapiro* [1958], instead of functions $u_\theta(\theta, \varphi)$ and $u_\varphi(\theta, \varphi)$ we must use the complex functions

$$u_+(\theta, \varphi) = u_\varphi(\theta, \varphi) + iu_\theta(\theta, \varphi),$$
$$u_-(\theta, \varphi) = u_\varphi(\theta, \varphi) - iu_\theta(\theta, \varphi) \qquad (28)$$

We shall expand these functions into the series

$$u_+(\theta, \varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Z^+_{lm} Y^l_{+m}(\theta, \varphi) = \sum_{l,m} Z^+_{lm} e^{im\varphi} P^l_{1m}(\cos\theta) \qquad (29)$$

$$u_-(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Z_{lm}^- Y_{-m}^l(\theta, \varphi)$$
$$= \sum_{l,m} Z_{lm}^- e^{im\varphi} P_{-1m}^l(\cos\theta) \tag{30}$$

where $P_{1m}{}^l(\cos\theta)$, $P_{-1m}{}^l(\cos\theta)$ are the functions introduced by Gelfand and collaborators which differ from the functions

$$\frac{dP_l^m(\cos\theta)}{d\theta} \mp \frac{m}{\sin\theta} P_l^m(\sin\theta)$$

only by numerical multipliers. It can easily be shown that for the homogeneous field $\mathbf{u}(\theta, \varphi)$ random coefficients $Z_{lm}{}^+$ and $Z_{lm}{}^-$ will satisfy the equations

$$\left.\begin{aligned} \overline{Z_{lm}^+ Z_{jn}^{+*}} &= \overline{Z_{lm}^- Z_{jn}^{-*}} = \delta_{lj}\,\delta_{mn} a_l \\ \overline{Z_{lm}^+ Z_{jn}^{-*}} &= \delta_{lj}\,\delta_{mn} c_l \end{aligned}\right\} \tag{31}$$

where $a_l \geq 0$ and c_l are real numbers such that $|c_l| \leq a_l$. Now with the help of (29) to (31) we can obtain the following formulas for the longitudinal and transverse correlation functions $B_{ll}(\theta)$ and $B_{nn}(\theta)$ of our vector field $u(\theta, \varphi)$:

$$B_{ll}(\theta) = \sum_{l=1}^{\infty} \left[\Phi_l^{(1)} \frac{d^2 P_l(\cos\theta)}{d\theta^2} + \Phi_l^{(2)} \frac{1}{\sin\theta} \frac{dP_l(\cos\theta)}{d\theta} \right] \tag{32}$$

$$B_{nn}(\theta) = \sum_{l=1}^{\infty} \left[\Phi_l^{(1)} \frac{1}{\sin\theta} \frac{dP_l(\cos\theta)}{d\theta} + \Phi_l^{(2)} \frac{d^2 P_l(\cos\theta)}{d\theta^2} \right] \tag{33}$$

where

$$\Phi_l^{(1)} = \frac{2l+1}{2l(l+1)} (a_l + c_l)$$

and

$$\Phi_l^{(2)} = \frac{2l+1}{2l(l+1)} (a_l - c_l)$$

are nonnegative constants. Conversely, any functions of the form (32) and (33) with nonnegative $\Phi_l{}^{(1)}$ and $\Phi_l{}^{(2)}$ can be longitudinal and transverse correlation functions of some homogeneous random vector field over the sphere.

The results (29) to (33) can also be obtained with the help of the transition from the components $u_\theta(\theta, \varphi)$ and $u_\varphi(\theta, \varphi)$ to the correlated scalar velocity potential and stream function and the following expansion of the two scalar functions in the series of the form of (27).

7. Atmospheric turbulence in the conditions of free convection is a very important geophysical example of nonhomogeneous turbulence; see *Obuchov* [1960]. Here we are dealing with the turbulence in the semispace $z \geq 0$ (where $z = 0$ is the surface of the earth), which is invariant to all translations, rotations, and reflections in the plane Oxy, but which is not axisymmetrical. If we disregard the thin layer of air near the earth surface and consider only the points of observation with the mutual distances greater than the internal scale of the turbulence, we can neglect the friction and heat conductivity terms from the hydrodynamical and thermodynamical equations; then the full system of equations becomes invariant to all similarity transformations of the form

$$x \to kx \quad y \to ky \quad z \to kz \qquad k > 0 \tag{34}$$

(see *Obuchov* [1960]). More precisely, the equations of free convection permit similar solutions of the form

$$\left.\begin{aligned} u_i(\mathbf{x}) &= z^\alpha u_i'(\mathbf{x}) \\ p(\mathbf{x}) &= z^{2\alpha} p'(\mathbf{x}) \\ T(\mathbf{x}) &= z^{2\alpha-1} T'(\mathbf{x}) \end{aligned}\right\} \tag{35}$$

where $\mathbf{x} = (x, y, z)$, and $u_i(\mathbf{x})$, $p(\mathbf{x})$, and $T(\mathbf{x})$ are the velocity, pressure, and temperature at a point $\mathbf{x}$ of fluid, and $u_i'(\mathbf{x})$, $p'(\mathbf{x})$, and $T'(\mathbf{x})$ are new random fields invariant to all translations, rotations, and reflections in the plane Oxy and to all extensions (34) (from dimensional considerations it is easy to see that $\alpha = 1/3$). Then correlation functions of the fields u_i', p', and T' and longitudinal and transverse correlation functions of the vector field (u_1', u_2') will depend only on two arguments z''/z' and r/z', where r is the horizontal distance between the points of observation.

Now the general form of the positive definite functions $B(z''/z', r/z')$ can be obtained with the help of the general theory given in *Yaglom* [1961]. Let us consider the simpler case of a two-dimensional random field in the semiplane $-\infty < x < \infty$, $0 < z < \infty$ invariant to all translations and reflection of the form $x \to x + a$, $z \to z$ and $x \to -x$, $z \to z$ and to extensions $x \to kx$, $z \to kz$ with $k > 0$. The corresponding correlation function also will have the form of the function $B(z''/z', r/z')$ of two variables (where $r = |x'' - x'|$). The general form of this function can be easily obtained from the results of *Gelfand and Naimark* [1947]. Here

$$B\left(\frac{z''}{z'}, \frac{r}{z'}\right) = \int_{-\infty}^{\infty} e^{ik \log z''/z'} \Phi(k)\, dk + \sum_{j,k=1}^{\infty} H_{jk}\left(\frac{z''}{z'}, \frac{r}{z'}\right)\Phi_{jk} \quad (36)$$

where

$$H_{jk}(\xi, \eta) = \frac{(-1)^j}{j!}\sqrt{\xi} \cdot \left\{\frac{d^j}{dt^j}\left[\frac{(\xi - i\eta - t)^k(1+t)^j}{(\xi + i\eta + t)^{k+1}} + \frac{(\xi + i\eta - t)^k(1+t)^j}{(\xi - i\eta + t)^{k+1}}\right]\right\}_{t=1} \quad (37)$$

$\Phi(k)$ is an arbitrary nonnegative function, and $||\Phi_{jk}||$ an arbitrary infinite nonnegative definite matrix, that is, the set of numbers Φ_{jk} such that

$$\text{Det } ||\Phi_{jk}||_1^n \geq 0$$

for all $n = 1, 2, 3 \cdots$. The different correlation functions are then defined by the choice of the nonnegative function $\Phi(k)$ and the infinite nonnegative definite matrix Φ_{jk}. The general form of the correlation function $B(z''/z', r/z')$ in the three-dimensional case is the same but with different functions $H_{jk}(\xi, \eta)$. Using (36) we can also obtain the analogue of the spectral representation for random fields describing atmospheric turbulence under conditions of free convection; but this will be given in another article.

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