On the Eulerian-Lagrangian Transform in the Statistical Theory of Turbulence[1]

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Abstract. Two important types of probing of a turbulent velocity field $\mathbf{u}(\mathbf{r}, t)$ are the Eulerian probings defined by $d\mathbf{r}/dt = \mathbf{v}$ ($\mathbf{v}$ constant) and the Lagrangian probing defined by $d\mathbf{r}/dt = \mathbf{u}(\mathbf{r}, t)$. Explicit expressions are derived for the transformation of autocorrelations and power spectra obtained by Eulerian and Lagrangian probing in the case of fully developed isotropic and homogeneous turbulence. The derivations are based on a statistical representation of the turbulent velocity field using the results of the equilibrium theory of turbulence. The Taylor hypothesis is verified in the limit of high probing velocities. The Hay-Pasquill conjecture relating the Lagrangian and Eulerian power spectra results as an approximation to the transformation equations. Application of the results to the theory of turbulent diffusion is indicated.

Introduction. It is the aim of the present investigation to establish the relation between the Eulerian and Lagrangian correlations in the case of fully developed isotropic homogeneous turbulence.

In the reference frame in which the average velocity of the fluid is zero, Eulerian and Lagrangian probings of a fluctuating velocity field denoted by

$$\mathbf{u} = \mathbf{u}(\mathbf{r}, t) \tag{1}$$

are characterized by the equations

$$d\mathbf{r}/dt = \mathbf{v} \qquad \text{(Euler)}$$

and

$$d\mathbf{r}/dt = \mathbf{u}(\mathbf{r}, t) \qquad \text{(Lagrange)} \tag{2}$$

where $\mathbf{v}$ is a constant velocity.

The two types of probing in general lead to different functional forms of the scalar autocorrelation coefficient defined by

$$R(\tau) = \overline{\mathbf{u}(t)\cdot\mathbf{u}(t+\tau)}/3u'^2 \tag{3}$$

where u' is the rms of any single component of $\mathbf{u}$.

The statistical model. In formulating the statistical model to be used here we shall be guided partly by the statistical theory of shot effect noise [*Rice*, 1944, 1945] and partly by the Helmholtz theorems.

We shall assume that the random part of the velocity field in fully developed turbulence may be written as a sum

$$\mathbf{u}(\mathbf{r}, t) = \sum_i \mathbf{F}_i(\mathbf{r} - \mathbf{R}_i) \tag{4}$$

over a very large number of small overlapping disturbances. Each disturbance corresponds to one of the many degrees of freedom in the turbulent velocity field and is characterized by its position $\mathbf{R}_i(t)$ in space and a number of time-independent parameters describing the functional form of $\mathbf{F}_i$. To ensure homogeneity we require that at any instant the distribution of disturbance positions is random and that their number within unit volume is Poisson-distributed with an average μ. Isotropy is ensured by requiring the distribution of orientations of the $\mathbf{F}_i$'s to be isotropic, and finally we shall obtain stationariness by assuming the distributions of all parameters to be time independent.

To comply with Helmholtz theorems we shall further assume that the disturbances participate in the fluid motion and shall express it by the equations

$$\mathbf{U}_i = \frac{d\mathbf{R}_i}{dt} = \sum_k \mathbf{F}_k(\mathbf{R}_i - \mathbf{R}_k) \tag{5}$$

We shall briefly justify the above assumptions using arguments from the equilibrium theory of

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turbulence. In this theory the turbulence may be characterized by the size of the energy-containing eddies λ_e and the rms of the velocity u'. Defining the Reynolds number by $Re \sim \lambda_e u'/\nu$, the dissipation scale is given by $\lambda_d \sim \lambda_e Re^{-3/4}$ and the number of degrees of freedom per unit volume is given approximately by $\mu \sim \lambda_d^{-3} \sim \lambda_e^{-3} Re^{9/4}$; we note that this is a very large number, which will be used later as an expansion parameter.

Assuming the average disturbances $\mathbf{F}_i$ to have a range λ, we may estimate the velocity fluctuations within a disturbance from Kolmogoroff-Obukhov's law to be less than $u_\lambda' \sim u'(\lambda/\lambda_e)^{1/3}$. This may be used to estimate λ in this theory, since the fluctuating part of the velocity will be given by the uncorrelated sum of all disturbances. In this way we obtain the inequalities

$$\lambda > \lambda_e Re^{-27/44} \gg \lambda_d \tag{6}$$

from which a few important conclusions may be drawn.

First, we realize that since $\lambda \gg \lambda_d$ it is permissible to neglect viscosity and use the Helmholtz theorems when deriving the equation of motion for $\mathbf{R}_i$. Second, we see that there is indeed a strong overlap of disturbances, since the number within an average range is $N \sim \mu\lambda^3 > Re^{18/44}$. Finally, we see that it is permissible to neglect the time dependence of the parameters describing the shape of a disturbance, since the velocity of deformation is smaller than $u_\lambda' \sim u' Re^{-9/44}$, which is small compared with the average translational velocity u'.

The above considerations may also be used to estimate another expansion parameter

$$\epsilon = \overline{k^{2n-2m}} \cdot \overline{k^{2m}} / \overline{k^{2n}} \sim Re^{-1/2} \qquad (n > m > 0) \tag{7}$$

where averages are carried out over a spectrum $E(k)$ obeying the Kolmogoroff power law in the range $\lambda_e^{-1} < k < \lambda_d^{-1}$ and having some kind of cutoff at $k \sim \lambda_d^{-1}$.

The Eulerian-Lagrangian transformation. To bring forward the essential points without too much complication in notation we shall first consider the one-dimensional case, neglecting for simplicity the motion of the disturbances. In this simple case expression 9 for the Eulerian correlation $R_E(\tau)$ may easily be obtained by inserting the trivial identity

$$\overline{\left(\frac{d^n u}{dt^n}\right)_E^2} = \overline{\left[\left(v\frac{d}{dr}\right)^n u\right]^2} = v^{2n}\overline{\left(\frac{d^n u}{dr^n}\right)^2} = v^{2n}\overline{k^{2n}}u'^2 \tag{8}$$

into the Taylor expansion for $R_E(\tau)$

$$R_E(\tau) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{u'^2} \overline{\left(\frac{d^n u}{dt^n}\right)_E^2} \frac{\tau^{2n}}{(2n)!} = \int_0^\infty dk E(k) \cos(v\tau k) \tag{9}$$

where the summation has been performed under the integral over k.

Using the statistical model defined above, we can explicitly express a mixed moment or expectation value

$$\overline{\prod_m (d^m u/dr^m)^{p_m}}$$

of any product of u and its space derivatives in terms of a polynomium in μ with coefficients containing semi-invariants of a type defined by

$$\langle \prod_m [m]^{\tau_m} \rangle = \left\langle \int_{-\infty}^{\infty} dr \prod_m \left(\frac{d^m F_i}{dr^m}\right)^{\tau_m} \right\rangle_i \tag{10}$$

where $\langle\ \rangle_i$ indicates averaging over all parameters in F_i.

Two important special cases are

$$\overline{\left(\frac{d^n u}{dr^n}\right)^2} = \mu\langle [n]^2 \rangle = \overline{k^{2n}}u'^2 \tag{11}$$

and

$$\overline{u^{2n}} \simeq \frac{(2n)!}{2^n n!} \mu^n \langle [o]^2 \rangle^n = \frac{(2n)!}{2^n n!} u'^{2n} \tag{12}$$

The first expression may be used together with (7) to establish the relation

$$\epsilon = \frac{\langle [n-m]^2 \rangle \langle [m]^2 \rangle}{\langle [n]^2 \rangle \langle [o]^2 \rangle} \qquad (n > m > 0) \tag{13}$$

while the second expression, where only the dominant term in the expansion in μ has been retained, demonstrates that the model yields a Gaussian distribution for u in the limit $Re \to \infty$.

Using the above-mentioned expression for the mixed moments and under the same simplifying assumptions used in deriving the Eulerian correlation above, it is now possible to obtain the following expansion for the Lagrangian average

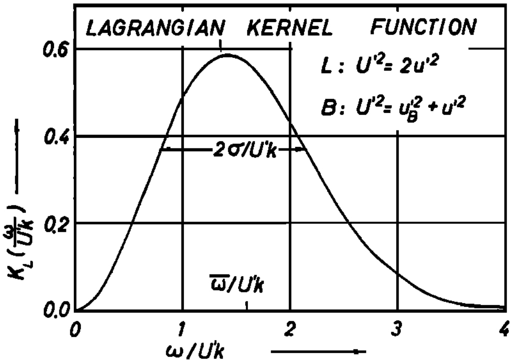

Fig. 1. The universal Lagrangian kernel function $K_L(\omega/U'k)$ as a function of $\omega/U'k$. Mean and variance of ω are given by $\bar{\omega}/U'k = \sqrt{8/\pi} = 1.60$ and $\sigma/U'k = 0.67$. Case L corresponds to Lagrangian probing. Case B corresponds to probing with a 'balloon' moving with a random velocity $\mathbf{u}_B$ with components distributed according to a Gaussian with rms u'_B.

of $(d^n u/dt^n)^2$ in terms of μ, ϵ, and the semi-invariants

$$\overline{\left(\frac{d^n u}{dt^n}\right)_L^2} = \overline{\left[\left(u\frac{d}{dr}\right)^n u\right]^2} = \frac{(2n)!}{2^n n!}\mu^{n+1}$$

$$\cdot\langle[o]^2\rangle^n\langle[n]^2\rangle \times \Bigg\{1 + \epsilon R(n)$$

$$+ \frac{1}{\mu}\left[n\frac{\langle[n]^2[o]^2\rangle}{\langle[n]^2\rangle\langle[o]^2\rangle}\right.$$

$$\left. + \frac{n(n-1)}{3!}\frac{\langle[o]^4\rangle}{\langle[o]^2\rangle^2} + \cdots\right]$$

$$+ \frac{1}{\mu^2}[\qquad] + \cdots\Bigg\}$$

$$\simeq \frac{(2n)!}{2^n n!}u'^{2n}\cdot\overline{k^{2n}}u'^2 \qquad (14)$$

where $R(n)$ is a rational algebraic function of n. The last equality is obtained in the limit $Re \rightarrow \infty$. Inserting this result in the Taylor expansion of $R_L(\tau)$, we obtain after performing the summation under the integral

$$R_L(\tau) = \int_0^\infty dkE(k)\exp\left[-\tfrac{1}{2}u'^2\tau^2k^2\right] \qquad (15)$$

Thus, under the above simplifying assumptions the Lagrangian correlation is obtained by transforming the energy spectrum with a Gaussian kernel, whereas the Eulerian correlation is obtained in the usual way by taking its cosine transform.

Three-dimensional formulation with moving disturbance centers. The procedure just described may also be applied to the three-dimensional scalar averages (3), taking into account the equation of motion (5), for the disturbance centers. A careful evaluation yields the following result for the Lagrangian average of $(d^n\mathbf{u}/dt^n)^2$

$$\overline{\left(\frac{d^n\mathbf{u}}{dt^n}\right)_L^2} \simeq \overline{\Big\{\sum_i[(\mathbf{u}-\mathbf{U}_i)\cdot\nabla]^n\mathbf{F}_i\Big\}^2}$$

$$\simeq \overline{|\mathbf{u}-\mathbf{U}|^{2n}}\cdot\overline{k^{2n}}\,\overline{u^2} \qquad (16)$$

where

$$\overline{|\mathbf{u}-\mathbf{U}|^{2n}} = \frac{(2n+2)!}{2^{n+1}(n+1)!}(2u'^2)^n \qquad (17)$$

since both $\mathbf{u}$ and $\mathbf{U}$ have the same Gaussian distribution for each of their components. The moment $\overline{k^{2n}}$ should in the three-dimensional case be taken over the energy spectrum

$$E(k) = \frac{2}{\pi}\int_0^\infty dr\left[\tfrac{1}{3}f(r) + \tfrac{2}{3}g(r)\right]\cos(kr) \qquad (18)$$

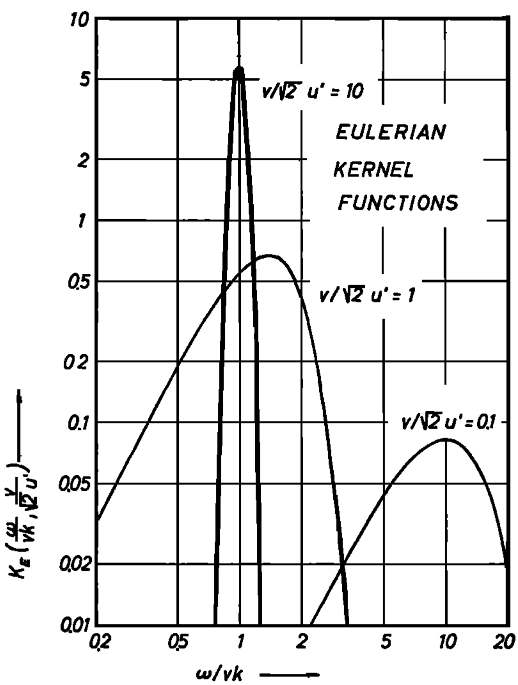

Fig. 2. Examples of Eulerian kernel functions for $v/\sqrt{2}\,u' = 0.1$, 1, and 10.

where f and g are the usual longitudinal and transverse spatial correlation coefficients.

Insertion of the above result into the Taylor expansion of $R_L(\tau)$ and performance of the summation under the integral yields the relation

$$R_L(\tau) = \int_0^\infty dkE(k) \cdot\left\{-\frac{d^2}{dz^2} \exp\left[\frac{-z^2}{2}\right]\right\}_{z=\sqrt{2}u'\tau k} \quad (19)$$

Similarly a relation may be derived for the Eulerian correlation obtained with a constant probing velocity v

$$R_E(\tau) = \frac{u'}{v} \int_0^\infty dkE(k) \cdot\left\{\frac{\partial}{\partial z}\left[\exp\left(\frac{-z^2}{2}\right) \sin\left(\frac{v}{u'} z\right)\right]\right\}_{z=u'\tau k} \quad (20)$$

Applications and discussion. Using the well-known relation

$$P(\omega) = \frac{2}{\pi} \int_0^\infty d\tau R(\tau) \cos \omega\tau \quad (21)$$

we may now express the Lagrangian and Eulerian power spectra as transforms of the energy spectrum using corresponding characteristic kernel functions. In the Lagrangian case we obtain

$$P_L(\omega) = \int_0^\infty dk\, E(k) \cdot\left[K_L\left(\frac{\omega}{\sqrt{2}\, u'k}\right)/\sqrt{2}\, u'k\right] \quad (22)$$

where the universal Lagrangian kernel function

$$K_L(z) = \sqrt{2/\pi}\, z^2 \exp(-z^2/2) \quad (23)$$

has the form shown in Figure 1 with mean value $\bar{z} = \sqrt{8/\pi} = 1.60$ and variance $\overline{z^2} - \bar{z}^2 = 0.67$. If we approximate $K_L(z)$ by a δ function, we obtain the rough but simple relationship

$$P_L(\omega) \simeq \frac{\sqrt{\pi}}{4} \frac{1}{u'} E\left(\frac{\sqrt{\pi}}{4} \frac{\omega}{u'}\right) \quad (24)$$

The analogous transformation for the Eulerian power spectrum has the form

$$P_E(\omega) = \int_0^\infty dkE(k) \cdot\left[K_E\left(\frac{\omega}{vk}, \frac{v}{\sqrt{2}\, u'}\right)/u'k\right] \quad (25)$$

where the Eulerian kernel function is given by

$$K_E(z, y) = \frac{yz}{\sqrt{\pi}} \{\exp[-y^2(z-1)^2] - \exp[-y^2(z+1)^2]\} \quad (26)$$

In Figure 2 the shape of this function is shown for $y = v/\sqrt{2}\, u'$ equal to 0.1, 1, and 10. It is seen how the function approaches the δ function, $\delta(z-1)$, for large values of y, yielding in the limit the relation

$$P_E(\omega) = E(\omega/v)/v \quad (27)$$

which is equivalent to Taylor's hypothesis. In Figure 3 the average value and the variance of the Eulerian kernels has been drawn in the range $0.1 \leq y \leq 10$.

Combining the above simple relations for the power spectra, we obtain as a first rough approximation in the case of smooth energy spectra the relation conjectured by *Hay and Pasquill* [1960]

$$P_L(\omega) \simeq \beta P_E(\beta\omega) \quad \text{with} \quad \beta \simeq \frac{\sqrt{\pi}}{4} \frac{v}{u'} \quad (28)$$

Inserting the Lagrangian correlation derived above in the well-known *Taylor* [1921] equation

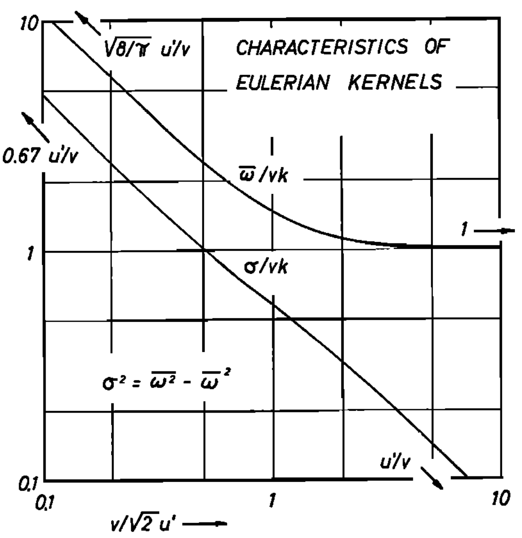

Fig. 3. Mean value and variance of Eulerian kernels in dimensionless units as a function of $v/\sqrt{2}\,u'$.

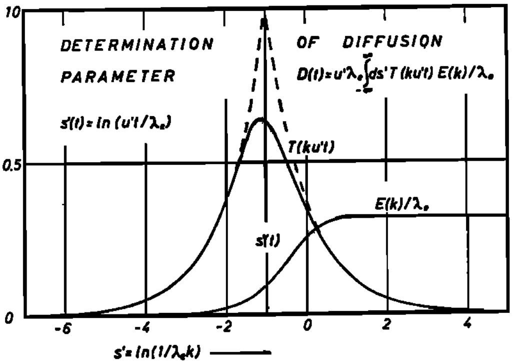

Fig. 4. Plot for determination of the diffusion parameter

$$D(t) = u' \int_{-\infty}^{\infty} ds\, E(k)T(u'tk)$$

The function T moves without change of shape toward larger s values with a speed proportional to ln t. $s' = s - \ln \lambda_0$.

describing turbulent diffusion, we obtain for the rms displacement r' the relation

$$r'^2(t) = D(t) \cdot t \qquad (29)$$

where the time-dependent diffusion parameter D is given by the following integral in which the substitution $s = \ln (l/k)$ has been made:

$$D(t) = u' \int_{-\infty}^{\infty} dsE(e^{-s})T(e^{-s+\ln(u't)}) \qquad (30)$$

The diffusivity kernel function T has the form

$$T(z) = [1 - \exp(-z^2)]/z \qquad (31)$$

The above expression for the diffusion parameter facilitates both the practical calculation and the intuitive understanding of turbulent diffusion problems. The function T may be approximated for large values of $|s - \ln(u't)|$ by the simple function $\exp[-|s - \ln(u't)|]$. By plotting T and the spectrum E as a function of s, i.e., on a logarithmic wave-number scale, as has been done in Figure 4, it is seen how the shape of the energy spectrum will affect the time dependence of D. The kernel function T advances without change of shape toward larger scales with an abscissa proportional to ln t, and gives rise to enhanced diffusion in the initial stage by making $D(t)$ proportional to t. Later, when passing into the region where $E(k)$ is flat, $D(t)$ becomes almost constant, and we obtain Fickian diffusion. The theory thus covers completely the range of Sutton's diffusion formula. Further applications and extensions of this theory to turbulent diffusion will be discussed by Kofoed-Hansen in a separate contribution to this symposium. A more detailed account of the present work will be published in a forthcoming Risø Report.

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