On the Problem of Formulating a Realistic Model of the General Atmospheric Circulation[1]

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Abstract. Since the equations of global mean flow and mean temperature (averages taken over a period of a year, say) are very complex in form when expressed in spherical polar coordinates, a limited problem is first considered: given the mean temperature and density distributions, we attempt to solve the residual velocity distribution equations. The Reynolds stress component associated with the meridional flow of zonal velocity is a crucial factor, and, as a first approximation, it is taken to depend linearly on Ω, the earth's angular velocity, and on the meridional mean temperature gradient. A series solution for the velocity distribution is then obtained in terms of ascending powers of Ω, and the meridional component series is used to explore the crucial influence of the Reynolds stress components on the structure of the meridional cellular circulation.

1. *Introduction.* In this paper the atmosphere is thought of as a rotating, thin, spherical shell of air in a turbulent flow, whose energy is derived from the sun's heating, dominant parameters being the mean temperature difference between equator and pole and the earth's angular velocity. The day-to-day changes in flow are chaotic, but when mean values are taken over periods of 6 months or a year a regular mean flow pattern is revealed. To describe this process quantitatively the equations of motion and of heat transfer, expressed in spherical polar coordinates, are averaged over a suitable period of time by means of the usual Reynolds technique, the resulting equations representing a mean flow associated with averaged temperature and heating distributions appropriate for a 1-year or 6-month period; the averaging process is also carried out around each latitude circle, although ultimately it is believed to be not beyond the scope of mathematical analysis, with the aid of electronic computation, to incorporate a longitudinal effect due to the difference in flow over continent and sea.

The nonlinear inertia terms and the terms describing the turbulence effects are of course very complex in form; in a first approach, only the nonlinear term involving the square of the zonal velocity is included and a boundary-layer form of approximation is employed, so that vertical gradients of velocity are retained and meridional gradients of velocity neglected. At this stage in developing a technique for solving the equations of motion in spherical polar coordinates it is also found necessary to neglect ρ_*/ρ_0 by comparison with unity in terms involving $(1 + \rho_*/\rho_0)$, where ρ_0 is some suitable reference density and ρ_* the deviation of density from this value. The eddy viscosity technique, used successfully in certain aspects of micrometeorology and in aerodynamic studies of a turbulent boundary layer, is also employed here to express the Reynolds stresses, associated with vertical transport of zonal and meridional velocity, in terms of mean velocity gradients, the numerical values used for the eddy viscosity coefficients being extracted from available observational results. The Reynolds stress component associated with meridional flow of zonal velocity must be a crucial factor; we suggest that it is primarily *thermally* driven, and, as a first approximation, we take it to depend linearly on Ω, the earth's angular velocity, and on the meridional mean temperature gradient. A solution of the full feedback atmospheric problem (described by simultaneous mean velocity and heat transfer equations) for a given input of solar heating has not been attempted here, but a

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limited aspect is considered instead. Convenient empirical expressions for the averaged density and temperature distributions are substituted into the mean flow equations to test various hypotheses by checking whether the ensuing equations, when they are solved, yield a reasonably realistic description of the flow. In this way a base can be established from which we can develop more incisive models involving solutions of the thermodynamic equation also.

2. *The formulation of equations describing the general circulation.* Taking r and θ to denote distance from the center of the spherical shell of air and colatitude respectively, denoting meridional, zonal, and vertical velocity components by V, U, W, pressure by p, density by ρ, and gravitational attraction on unit mass near the earth's surface by g', and neglecting all viscous stresses, since Reynolds eddy stresses are very much larger, the equations of zonally uniform motion are

$$\rho\left(\frac{\partial V}{\partial t} + W\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \theta} - \frac{U^2 \cot\theta}{r} + \frac{VW}{r}\right) = -\frac{\partial p}{r\,\partial\theta} \quad (1)$$

for meridional flow,

$$\rho\left(\frac{\partial U}{\partial t} + W\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \theta} + \frac{UW}{r} + \frac{UV\cot\theta}{r}\right) = 0 \quad (2)$$

for zonal flow, and

$$\rho\left(\frac{\partial W}{\partial t} + W\frac{\partial W}{\partial r} + \frac{V}{r}\frac{\partial W}{\partial \theta} - \frac{(U^2 + V^2)}{r}\right) = -\frac{\partial p}{\partial r} - g'\rho \quad (3)$$

for vertical flow, and the associated equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 W) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta\, V) = 0 \quad (4)$$

It is convenient to treat the motion as a basic solid rotation flow, with the angular velocity Ω of the earth and uniform temperature, together with a flow relative to the rotating earth, and we write for the solid rotation flow $V = V_0 = 0$, $U = U_0 = \Omega r \sin\theta$, $W = W_0 = 0$, $\rho = \rho_0$, $p = p_0$. The equations describing the uniform rotation then become

$$-r\Omega^2 \sin^2\theta = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial r} - g' \quad (5)$$

and

$$-r\Omega^2 \cos\theta\sin\theta = -\frac{1}{\rho_0}\frac{\partial p_0}{r\,\partial\theta} \quad (6)$$

and writing $V = v$, $U = \Omega r \sin\theta + u$, $W = w$, $p = p_0 + p_*$, $\rho = \rho_0 + \rho_*$, the equations of flow relative to the rotating earth become

$$\left(1 + \frac{\rho_*}{\rho_0}\right)\left(\frac{\partial v}{\partial t} + w\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} - \frac{u^2}{r}\cot\theta + \frac{vw}{r} - 2\Omega u\cos\theta\right) = \frac{\rho_*}{\rho_0} r\Omega^2 \sin\theta\cos\theta - \frac{1}{\rho_0}\frac{\partial p_*}{r\,\partial\theta} \quad (7)$$

$$\left(1 + \frac{\rho_*}{\rho_0}\right)\left(\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial\theta} - \frac{u^2}{r} - \frac{v^2}{r} - 2\Omega u\sin\theta\right) = \frac{\rho_*}{\rho_0} r\Omega^2 \sin^2\theta = \frac{1}{\rho_0}\frac{\partial p_*}{\partial r} - g'\frac{\rho_*}{\rho_0} \quad (8)$$

and

$$\frac{\partial u}{\partial t} + w\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} + \frac{uw}{r} + \frac{uv}{r}\cot\theta + 2\Omega w\sin\theta + 2\Omega v\cos\theta = 0 \quad (9)$$

and the equation of continuity becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 w\left(1 + \frac{\rho_*}{\rho_0}\right)\right\} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta v\left(1 + \frac{\rho_*}{\rho_0}\right)\right\} = 0 \quad (10)$$

We now substitute, into equations 7, 8, 9, and 10, $u = \bar{u} + u'$, $v = \bar{v} + v'$, $w = \bar{w} + w'$, where $\bar{u}$, $\bar{v}$, $\bar{w}$ denote mean values over time periods long enough for $\overline{u'} = \overline{v'} = \overline{w'} = 0$, and it is *assumed* that these mean values are independent of successive averaging periods and that eddy fluctuations of ρ_* do not contribute significantly to the ensuing Reynolds stress system.

The continuity equation 10 is then valid with $\bar{u}$ and $\bar{w}$ written for u and w, and equation 7 becomes

$$\left(1+\frac{\rho_*}{\rho_0}\right)\left\{\bar{w}\frac{\partial\bar{v}}{\partial r}+\frac{\bar{v}}{r}\frac{\partial\bar{v}}{\partial\theta}-\frac{\bar{u}^2}{r}\cot\theta+\frac{\bar{w}\bar{v}}{r}-2\Omega\bar{u}\cos\theta+\left[\frac{\partial}{\partial r}(\overline{v'w'})+\frac{1}{r}\frac{\partial}{\partial\theta}(\overline{v'v'})+\frac{(\overline{v'v'}-\overline{u'u'})}{r}\cot\theta+\frac{3\overline{w'v'}}{r}+\overline{\frac{v'}{r^2}\frac{\partial}{\partial r}\left(w'r^2\frac{\rho_*}{\rho_0}\right)}+\overline{\frac{v'}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v'\sin\theta\frac{\rho_*}{\rho_0}\right)}\right]\right\}=\frac{\rho_*}{\rho_0}r\Omega^2\sin\theta\cos\theta-\frac{1}{\rho_0}\frac{\partial p_*}{r\,\partial\theta}\qquad(11)$$

and two similar equations are obtained for the mean flow in the zonal and vertical directions; they are seen to incorporate a complex Reynolds stress system, but many components are probably very small and negligible.

3. *The mathematical approximations, physical assumptions, and boundary conditions used in formulating the model.* To make the system of mean motion equations mathematically tractable two basic approximations are made: (1) Since the ratio of (tropospheric thickness)/(radius of the earth) is very small, the usual boundary-layer approximation is made, and consequently *vertical gradients* of velocity are retained, meridional gradients of velocity are neglected, shearing types of eddy stresses are assumed dominant, and certain terms involving $1/r$ are neglected. (2) ρ_*/ρ_0, a function of r and θ, is neglected by comparison with 1; taking ρ_0 at a reference point on the surface, it is found that ρ_*/ρ_0 varies between zero and about 2/5 at the tropopause, and so the error involved is probably small but increases as the tropopause is approached. The continuity equation 10 becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\bar{w})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\bar{v})=0\qquad(12)$$

and the equations of mean motion become

$$\bar{w}\frac{\partial\bar{v}}{\partial r}+\frac{\bar{v}}{r}\frac{\partial\bar{v}}{\partial\theta}-\frac{\bar{u}^2}{r}\cot\theta+\frac{\bar{w}\bar{v}}{r}-2\Omega\bar{u}\cos\theta=-\frac{1}{\rho_0}\frac{\partial p_*}{r\,\partial\theta}+\frac{\rho_*}{\rho_0}r\Omega^2\sin\theta\cos\theta-\frac{\partial}{\partial r}(\overline{v'w'})\qquad(13)$$

$$\bar{w}\frac{\partial\bar{u}}{\partial r}+\frac{\bar{v}}{r}\frac{\partial\bar{u}}{\partial\theta}+\frac{\bar{u}\bar{v}}{r}\cot\theta+\frac{\bar{w}\bar{u}}{r}+2\Omega\bar{v}\cos\theta+2\Omega\bar{w}\sin\theta=-\frac{\partial}{\partial r}(\overline{u'w'})-\frac{\partial}{r\,\partial\theta}(\overline{u'v'})\qquad(14)$$

and

$$\bar{w}\frac{\partial\bar{w}}{\partial r}+\frac{\bar{v}}{r}\frac{\partial\bar{w}}{\partial\theta}-\frac{\bar{u}^2}{r}-\frac{\bar{v}^2}{r}-2\Omega\bar{u}\sin\theta=-\frac{1}{\rho_0}\frac{\partial p_*}{\partial r}-g'\frac{\rho_*}{\rho_0}+\frac{\rho_*}{\rho_0}r\Omega^2\sin^2\theta\qquad(15)$$

The meridional gradient of $\overline{u'v'}$ is retained in equation 14, as it is this stress component that assists in driving angular momentum from low to high latitudes. It must be thermally driven by the temperature difference between equator and pole, and it plays a crucial part in the circulation. In various aspects of micrometeorology and aerodynamics the eddy viscosity formulation has been successfully employed in constructing a description of turbulent boundary layers: the work of *Phillips* [1956], employing Cartesian coordinates, also suggests that it may be used in formulating a general circulation model. We write then, for $\overline{u'w'}$ and $\overline{v'w'}$, the relations

$$\overline{v'w'}=-K_{mv}(\partial\bar{v}/\partial r)\qquad(16)$$

and

$$\overline{u'w'}=-K_{zv}(\partial\bar{u}/\partial r)\qquad(17)$$

where K_{mv} is associated with vertical mixing of meridional velocity and K_{zv} with vertical mixing of zonal velocity. As a first approximation, K_{mv} and K_{zv} are assumed to be independent of position, and $\overline{u'v'}$ is assumed to depend linearly on Ω and on the meridional mean temperature gradient $\delta T/r\,\delta\theta$; and we write

$$\overline{u'v'}=-k\Omega(\partial T/r\,\partial\theta)\qquad(18)$$

Since the gradient of mean temperature is from equator to pole, $\overline{u'v'}$ is negative everywhere, i.e., toward the pole, and is consequently in the correct direction. *Phillips* [1956] finds that magnitudes of about 10^9 cm sec units for K_{mv} and K_{zv} lead to realistic circulations, whereas the work of *Tucker* [1960] in analyzing observed velocity values suggests numerical values of about 10^6 cm^2 sec^{-1} for K_{zv}. We find that values of *about* 10^9 for K_{mv} and *about* 10^6 for K_{zv} lead to fairly realistic calculated meridional circulations. A mean value for $(k\Omega)$ in (18) is calculated to be about 10^{12} cm sec °C units by dividing values of $(\overline{u'v'})$, deduced by Tucker, by the meridional gradient of mean temperature. (During the symposium discussion, however, it became clear that the model can be improved by allowing k to vary with ζ so that the effect of meridional temperature gradient varying with ζ can be incorporated. We note also that the observed values of $(\overline{u'v'})$ could not be used directly, as they must depend on Ω, and we consider a solution in powers of Ω.)

A convenient empirical formula is now required for ρ_*/ρ_0 and T/T_0. The actual height of the tropopause varies from about 15 km at the equator to about 10 km at the pole, and, although we hope to extend the analysis to this practical case, we have assumed in our model that the height is independent of latitude. The analysis is then conveniently expressed in terms of the variable ζ defined by

$$r = a(1 + \zeta f) \tag{19}$$

where a denotes the earth's radius and $f = h/a$, h being the height of the troposphere, so that the value of ζ lies between 0 and 1 in the troposphere. A reasonably good empirical mean annual temperature formula (giving a zero vertical temperature gradient at the tropopause) is found to be

$$T/T_0 = A_0 + B_0(\zeta - \tfrac{1}{2}\zeta^2) + P_2(\cos\theta)\{A_2 + B_2(\zeta - \tfrac{1}{2}\zeta^2)\} \tag{20}$$

where $P_2(\cos\theta)$ is a Legendre function and T_0 a convenient reference temperature; higher powers in the ζ polynomials render the ensuing analysis unmanageable by a desk calculating machine. The numerical values of the coefficients are chosen to obtain a reasonably good fit to the available mean annual temperature distribution for the northern hemisphere. It is assumed that northern and southern hemispheric distributions coincide, but the analysis can be extended to apply to northern mean winter conditions, say, and southern mean summer conditions, or vice versa, by including several more Legendre-type terms, but in a first approach symmetry about the equator is assumed. To obtain a reasonably representative density formula, equation 20 is used together with the pressure equation

$$p = p_0 \exp\left[-(g/R)\int dr/T\right]$$

and the equation of state

$$p = \rho RT$$

where R is the gas constant, to obtain a working density expression in the form

$$\rho_*/\rho_0 = a_0 + b_0(\zeta - \tfrac{1}{2}\zeta^2) + P_2(\cos\theta)\{a_2 + b_2(\zeta - \tfrac{1}{2}\zeta^2)\} \tag{21}$$

Substituting equations 16, 17, 18, 19, 20, and 21 into 13, 14, and 15, and eliminating pressure between equations 13 and 15, we obtain

$$\begin{aligned}
&\left\{2\Omega h \sin\theta \frac{\partial \bar{u}}{\partial \theta} - 2\Omega h \frac{(1 + \zeta f)}{f} \cos\theta \frac{\partial \bar{u}}{\partial \zeta}\right\} \\
&+ \left\{\bar{w} \frac{(1 + \zeta f)}{f} \frac{\partial^2 \bar{v}}{\partial \zeta^2} + \bar{w} \frac{\partial \bar{v}}{\partial \zeta} \right. \\
&+ \frac{\partial}{\partial \zeta}\left(\bar{v} \frac{\partial \bar{v}}{\partial \theta} - \bar{u}^2 \cot\theta + \bar{v}\bar{w}\right) \\
&- \frac{\partial}{\partial \theta}\left(\bar{w} \frac{\partial \bar{w}}{\partial \zeta} + \frac{\bar{v} f}{(1 + \zeta f)} \frac{\partial \bar{w}}{\partial \theta} - \frac{\bar{v}^2 f}{(1 + \zeta f)} \right. \\
&\left.\left. - \frac{\bar{u}^2 f}{(1 + \zeta f)}\right) + \frac{(1 + \zeta f)}{f} \frac{\partial \bar{v}}{\partial \zeta} \frac{\partial \bar{w}}{\partial \zeta}\right\} \\
&= \left\{\frac{K_{mv}}{h} \frac{\partial^2 \bar{v}}{\partial \zeta^2} + \frac{K_{mv}}{hf}(1 + \zeta f) \frac{\partial^3 \bar{v}}{\partial \zeta^3}\right\} \\
&+ \left\{g'h \frac{\partial}{\partial \theta}\left(\frac{\rho_*}{\rho_0}\right) \right. \\
&+ a^2(1 + \zeta f)^2 \Omega^2 \sin\theta \cos\theta \frac{\partial}{\partial \zeta}\left(\frac{\rho_*}{\rho_0}\right) \\
&\left. - a(1 + \zeta f)\Omega^2 h \sin^2\theta \frac{\partial}{\partial \theta}\left(\frac{\rho_*}{\rho_0}\right)\right\}
\end{aligned} \tag{22}$$

The parameter f is very small (about 1/290), and the vertical velocity $\bar{w}$ is much smaller than $\bar{u}$ or $\bar{v}$; these factors enable us to neglect many of the terms in (22) and retain only the main terms within the various brackets. We obtain, after integration with respect to ζ,

$$\frac{\partial^2 \bar{v}}{\partial \zeta^2} + 2R_{mv} \cos \theta \bar{u} - \frac{h}{K_{mv}}\left(\bar{w}\frac{\partial \bar{v}}{\partial \zeta} + f\bar{v}\frac{\partial \bar{v}}{\partial \theta} - f\bar{u}^2 \cot \theta\right) = \frac{g'h^2 f}{K_{mv}}\frac{d}{d\theta} P_2(\cos \theta)\left\{a_2\zeta + b_2(\tfrac{1}{2}\zeta^2 - \tfrac{1}{6}\zeta^3)\right\} - \frac{a^2\Omega^2 hf}{K_{mv}} \sin \theta \cos \theta \times \{b_0 + b_2 P_2(\cos\theta)\}(\zeta - \tfrac{1}{2}\zeta^2) + F(\theta) \quad (23)$$

where $R_{mv} = \Omega h^2/K_{mv}$.

Similarly, the zonal equation 14 becomes

$$\frac{\partial^2 \bar{u}}{\partial \zeta^2} - 2R_{zv} \cos \theta \bar{v} + \left(\frac{\lambda h^2 R_{mv}}{a^2 K_{zv}}\right)\frac{\partial^2 T}{\partial \theta^2} = \frac{h}{K_{zv}}\left(\bar{w}\frac{\partial \bar{u}}{\partial \zeta} + f\bar{v}\frac{\partial \bar{u}}{\partial \theta} + f\bar{u}\bar{v} \cot \theta\right) \quad (24)$$

where $\lambda = k\Omega/R_{mv}$ and $R_{zv} = \Omega h^2/K_{zv}$. Equations 23 and 24 form the basic equations of motion to be solved: they represent a balance between the Coriolis forces, the turbulent stresses, the nonlinear inertia terms, and terms incorporating the effects of meridional and vertical temperature gradient.

The boundary conditions used are the simplest possible in form in order to reduce the weight of numerical work; they are

$$\bar{u} = \bar{v} = \bar{w} = 0 \quad \text{on} \quad \zeta = 0 \quad (25)$$

and

$$\frac{\partial \bar{u}}{\partial \zeta} = \frac{\partial \bar{v}}{\partial \zeta} = \frac{\partial \bar{w}}{\partial \zeta} = 0 \quad \text{on} \quad \zeta = 1 \quad (26)$$

A better form of prescribed conditions at the surface $\zeta = 0$ would be given by

$$K_{zv}(\partial \bar{u}/\partial \zeta) = \kappa \bar{u}^2, \qquad K_{mv}(\partial \bar{v}/\partial \zeta) = \kappa \bar{v}^2$$

that is, Reynolds stresses very near the surface are proportional to squares of velocities, κ being known empirically. However, this form complicates the solution very considerably, and, in a first approach, the simpler form (25) has been taken in obtaining a first solution of (23) and (24). Boundary conditions at the tropopause for a general circulation type of problem are not of course well established, and again the simplest assumption has been taken, that Reynolds stresses are zero at the tropopause.

4. *A solution of the equations of mean motion, including nonlinear terms.* Consideration of the magnitude of the various nonlinear terms in equation 23, using observed values, shows that the term involving $(\bar{u})^2$ is easily dominant, except in very low latitudes, and is of the same order of magnitude as the Coriolis term, and in equation 24 all the nonlinear terms are much smaller than the Coriolis term. Consequently, to estimate the influence of the nonlinear terms on the form of solution, we retain only the term involving $\bar{u}^2$. The parameter R_{mv} is nondimensional, and it is convenient to write $\bar{u} = K_1 u$, $\bar{v} = K_1 v$, where $K_1 = g'h^2/K_{mv}$ and has the dimensions of velocity; u and v are then nondimensional. Equations 23 and 24 become

$$\frac{\partial^2 v}{\partial \zeta^2} + 2R \cos \theta u + \left(\frac{g'h^3 f}{K_{mv}^2}\right)u^2 \cot \theta = -f\frac{d}{d\theta} P_2(\cos \theta)\{a_2\zeta + b_2(\zeta - \tfrac{1}{2}\zeta^2)\} - \left(\frac{a^2 f K_{mv}^2}{g'h^5}\right)R^2 \sin \theta \cos \theta\{b_0 + b_2 P_2(\cos \theta)\}(\zeta - \tfrac{1}{2}\zeta^2) + F(\theta) \quad (27)$$

where $R = R_{mv}$, and

$$\frac{\partial^2 u}{\partial \zeta^2} - 2R_{zv} \cos \theta v + \left(\frac{\lambda R K_{mv}}{g' a^2 K_{mz}}\right)\frac{\partial^2 T}{\partial \theta^2} = 0 \quad (28)$$

To obtain an estimate of the manner in which the effect of the nonlinear terms enters the flow, as the angular velocity Ω is assumed to increase from zero, we look for a solution of (27) and (28) in the form of an ascending series in the parameter R; later it is found that substitution of the value of the earth's angular velocity for Ω fortunately leads to reasonably quick convergence of the series solution for the meridional circulation over a substantial part of the atmosphere. We write

$$u = u_0 + Ru_1 + R^2u_2 + R^3u_3 + \cdots \quad (29)$$

$$v = v_0 + Rv_1 + R^2v_2 + R^3v_3 + \cdots \quad (30)$$

and

$$F(\theta) = F_0 + RF_1 + R^2F_2 + R^3F_3 + \cdots \quad (31)$$

Substituting (29), (30), and (31) into (27) and

(28), and equating the coefficients of R, R^2, etc., we obtain the following system of equations:

$$\frac{\partial^2 u_0}{\partial \zeta^2} = 0 \quad (32)$$

$$\frac{\partial^2 u_1}{\partial \zeta^2} = 2\frac{R_{zv}}{R_{mv}} \cos\theta v_0 - \left(\frac{\lambda K_{mv}}{g' a^2 K_{zv}}\right)\frac{\partial^2 T}{\partial \theta^2} \quad (33)$$

$$\frac{\partial^2 u_2}{\partial \zeta^2} = 2\frac{R_{zv}}{R_{mv}} \cos\theta v_1 \quad (34)$$

in general

$$\frac{\partial^2 u_n}{\partial \zeta^2} = 2\frac{R_{zv}}{R_{mv}} \cos\theta v_{n-1} \quad (35)$$

and also

$$\frac{\partial^2 v_0}{\partial \zeta^2} = -f\frac{d}{d\theta}P_2(\cos\theta)\{a_2\zeta + b_2(\tfrac{1}{2}\zeta^2 - \tfrac{1}{6}\zeta^3)\} - \left(\frac{g' h^3 f}{K_{mv}^2}\right)u_0^2 \cot\theta + F_0 \quad (36)$$

$$\frac{\partial^2 v_1}{\partial \zeta^2} + 2\cos\theta u_0 = -\left(\frac{2g' h^3 f}{K_{mv}^2}\right)\cot\theta u_0 u_1 + F_1 \quad (37)$$

$$\frac{\partial^2 v_2}{\partial \zeta^2} + 2\cos\theta u_1 = -\left(\frac{g' h^3 f}{K_{mv}^2}\right)\cot\theta(u_1^2 + 2u_0u_2) - \left(\frac{a^2 f K_{mv}^2}{g' h^5}\right)\sin\theta\cos\theta\{b_0 + b_2 P_2(\cos\theta)\}\times(\zeta - \tfrac{1}{2}\zeta^2) + F_2 \quad (38)$$

$$\frac{\partial^2 v_3}{\partial \zeta^2} + 2\cos\theta u_2 = -\left(\frac{g' h^3 f}{K_{mv}^2}\right)\cot\theta(u_1u_2 + u_0u_3) + F_3 \quad (39)$$

and so on.

The associated boundary conditions are $u_0 = v_0 = 0$ at $\zeta = 0$, since $u = v = 0$ independently of R, and similarly $\partial u_0/\partial\zeta = \partial v_0/\partial\zeta = 0$ at $\zeta = 1$. Equation 32 gives immediately $u_0 = 0$ and, hence, using (37), $v_1 = 0$; similarly, $u_2 = 0$ and $v_3 = 0$, etc. The zero value for u_0 shows of course that at $\Omega = 0$ the circulation is entirely meridional. Integration of equation 36 together with an application of the condition

$$\int_0^1 v_0\, d\zeta = 0$$

leads to the expression

$$v_0 = f\frac{d}{d\theta}P_2(\cos\theta)\left\{-\left(\frac{a_2}{8} + \frac{b_2}{24}\right)\zeta + \left(\frac{5a_2}{16} + \frac{b_2}{12}\right)\zeta^2 - \frac{b_2}{6}\zeta^3 - \frac{a_2}{24}\zeta^4 + \frac{b^2}{120}\zeta^5\right\} \quad (40)$$

When this is plotted as a function of ζ and θ, we find a meridional circulation in the form of a single Hadley cell, which is directed toward the south in lower altitudes and toward the north at higher altitudes.

Substitution of the v_0 expression and the empirical temperature form (20), with $\zeta = \frac{1}{2}$ to simplify the resulting polynomials, into (33) and integration then leads to

$$u_1 = -2f\left(\frac{K_{mv}}{K_{zv}}\right)\cos\theta\frac{d}{d\theta}P_2(\cos\theta)\cdot\left\{a_2\left(\frac{\zeta^3}{48} - \frac{5\zeta^4}{192} + \frac{\zeta^5}{120}\right) + b_2\left(\frac{\zeta^3}{144} - \frac{\zeta^4}{144} + \frac{\zeta^6}{720} - \frac{\zeta^7}{5040}\right)\right\} + 0.1725\left(\frac{K_{mv}\lambda}{K_{zv}a^2K_1}\right)\cos 2\theta(\zeta - \tfrac{1}{2}\zeta^2) \quad (41)$$

This is the dominating term if Ω is very small and on calculation gives an *easterly* flow at low latitudes and westerly elsewhere. Integration of (38) then yields the next term in the meridional series expansion,

$$v_2 = 4f\left(\frac{K_{mv}}{K_{zv}}\right)\cos^2\theta\frac{d}{d\theta}P_2(\cos\theta)\cdot\{a_2\times10^{-3}(1.389\zeta + 1.042\zeta^5 - 0.868\zeta^6 + 0.198\zeta^7) + b^2\times10^{-4}(-5.208\zeta + 3.472\zeta^5 - 2.315\zeta^6 + 0.248\zeta^8 - 0.028\zeta^9)\} - 0.345\left(\frac{K_{mv}\lambda}{K_{zv}a^2K_1}\right)\cos 2\theta\cdot\cos\theta\left(-\frac{\zeta}{3} + \frac{\zeta^3}{6} - \frac{\zeta^4}{24}\right)$$

$$- (0.1725)^2\left(\frac{h^3 g' f\lambda^2}{a^4 K_1^2 K_{zv}^2}\right) \cot\theta$$

$$\cdot \cos^2 2\theta\left(-\frac{2\zeta}{15} + \frac{\zeta^4}{12} - \frac{\zeta^5}{20} + \frac{\zeta^6}{120}\right)$$

$$- 4\left(\frac{h^3 g' f^3}{K_{zv}^2}\right) \cos^2\theta \cot\theta\left(\frac{d}{d\theta} P_2(\cos\theta)\right)^2$$

$$\cdot[a_2^2 \times 10^{-6}(-3.21\zeta + 7.75\zeta^8 - 15.07\zeta^9$$
$$+ 11.39\zeta^{10} - 3.95\zeta^{11} + 0.53\zeta^{12}) + 2a_2 b_2$$
$$\times 10^{-6}(-1.21\zeta + 2.58\zeta^8 - 4.52\zeta^9$$
$$+ 2.65\zeta^{10} - 0.26\zeta^{11} - 0.31\zeta^{12}$$
$$+ 0.11\zeta^{13} - 0.01\zeta^{14}) + b_2^2 \times 10^{-6}(-0.46\zeta$$
$$+ 0.86\zeta^8 - 1.34\zeta^9 + 0.54\zeta^{10} + 0.18\zeta^{11}$$
$$- 0.17\zeta^{12} + 0.02\zeta^{13} + 0.01\zeta^{14})]$$

$$- \left(\frac{K_{mv}^2}{fg' h^3}\right) \sin\theta \cos\theta(b_0 + b_2 P_2(\cos\theta))$$

$$\cdot\left(-\frac{\zeta}{3} + \frac{\zeta^3}{6} - \frac{\zeta^4}{24}\right) + \frac{F_2(\theta)}{K_1}(-\zeta + \tfrac{1}{2}\zeta^2)$$

$$+ 0.69\left(\frac{h^3 g' f\lambda}{a^2 K_1 K_{mv}^2}\right) \cot\theta$$

$$\cdot \cos^2\theta \frac{d}{d\theta} P_2(\cos\theta)[a_2 \times 10^{-4}(-6.2\zeta$$
$$+ 6.94\zeta^6 - 8.68\zeta^7 + 3.81\zeta^8$$
$$- 0.58\zeta^9) + b_2 \times 10^{-4}(-2.33\zeta + 2.32\zeta^6$$
$$- 2.48\zeta^7 + 0.62\zeta^8 + 0.19\zeta^9$$
$$- 0.10\zeta^{10} + 0.01\zeta^{11})] \qquad (42)$$

where $F_2(\theta)$ is obtained by substituting (42) into the condition

$$\int_0^1 v_2 \, d\zeta = 0$$

Substituting (42) into (35), with $n = 3$, gives the next term in the series for zonal velocity, and the procedure has been repeated to obtain expressions for u_3 and v_4. They are similar in structure to (42) but very much longer; if it is assumed that $K_{mv} = K_{zv}$ they contain a dominant term and tractable expressions can be obtained for u_5, v_6, and u_7, but if (K_{mv}/K_{zv}) is significantly greater than 1 the procedure becomes too cumbersome and breaks down.

5. *Discussion of numerical results.* The form of the equations and expressions for successive u_n and v_n illustrate clearly the dependence of the mean flow on the Reynolds stress components, but the choice of appropriate numerical values of the associated turbulence parameters K_{mv}, K_{zv}, and λ is of course a fundamental difficulty. We have taken the value 10^6 cm² sec⁻¹ for K_{zv}, since this was approximately the value suggested by *Tucker* [1960]. If we take the same value for K_{mv} and the associated λ value from Tucker's values of $\overline{u'v'}$, the meridional velocity components are of much smaller magnitude than observed values, and a calculated Hadley cell extending from the equator to the pole is produced. However, since the large-scale turbulence in the main body of the troposphere is generated by the temperature gradient between equator and pole, it seems reasonable to suppose that the eddy viscosity coefficient is of very much greater magnitude in the direction of this gradient than in the zonal direction. No observational data are available to support this suggestion, but we find that a fairly realistic meridional cellular structure is calculated if we substitute into the series solution a numerical value for K_{mv} of about 10^9 cm² sec⁻¹ together with the associated λ value which produces values of $\overline{u'v'}$ in rough correspondence with those given by Tucker. The form of the v_n terms becomes rapidly more complex as n increases, but fortunately it is only necessary to calculate v_0, v_2, and v_4, since the v_4 values are considerably smaller than the v_2 values except in very high latitudes.

The values corresponding to $\zeta = 1/2$ and $\zeta = 1$ are shown in Table 1. The breakdown of a

TABLE 1. Calculated Values of Mean Annual Meridional Velocity

Velocities in centimeters per second; positive indicates northerlies; $K_{zv} = 10^6$ cm² sec⁻¹; $K_{mv} = 10^9$ cm² sec⁻¹; $\lambda = 1.21 \times 10^{17}$ in cm sec °C units.

	$\zeta = \frac{1}{2}$		$\zeta = 1$	
Colatitude	Observed (Tucker)	Calculated	Observed (Tucker)	Calculated
20	−25	−1	+5	+1
30	−14	+5	−3	+4
40	+28	+9	+11	−15
50	+19	+8	+10	−26
60	0	+7	−30	−31
70	−35	+6	−58	−27
80	−3	+3	−48	−16

single Hadley cell is clearly marked at about latitude 55°, and during the calculation it was interesting to note that if λ is decreased there is a tendency for the circulation to degenerate into a single cell, whereas an increase in λ moves to lower latitudes the calculated point of breakdown into a second cell. This suggests, as we might expect, that the Reynolds stress component $\overline{u'v'}$ is a crucial factor in shaping the structure of the meridional cellular circulation. From equation 28 we see that $u_1, v_2, \cdots$, depend on $(\partial/\partial\theta)\,(\overline{u'v'})$; this changes sign at about 45° latitude, and since it makes a major contribution to v (and probably u) a reversal in sign of v takes place (at a value of θ that depends in our calculation on the value selected for λ) and so produces a double cell system instead of a system that is a simple Hadley cell in the absence of $\overline{u'v'}$. The K_{mv}/K_{sv} value is also found to influence the point of cellular division.

We have selected the terms in the resultant expression for v that depend on $\overline{u'v'}$ and those that depend on the nonlinear terms. We find that the contribution of these $\overline{u'v'}$ terms to the resultant v value is about 20 per cent at 20° latitude, 10 per cent at 40°, and 50 per cent at 60°; the contribution of nonlinear terms is about 2 per cent at 20° latitude, 5 per cent at 40°, and 10 per cent at 60°.

It has also been found possible to obtain sufficiently rapid convergence of the u series from latitude 10° to 40° using $K_{mv} = K_{sv} = 2 \times 10^7$ cm^2 sec^{-1}, and we find calculated u values corresponding fairly well with observed values, including the change from the westerlies of mid-latitudes to the easterlies of low latitudes. Unfortunately, however, the convergence of the u series is very slow for the values of K_{sv} and λ based on observed estimates of stresses and velocity gradients and the value of K_{mv} which leads to fairly realistic meridional circulation. The numerical evaluation using a desk calculator becomes quite impossible, owing to the rapidly increasing complexity of the successive u_n as n increases, but it is hoped to overcome this difficulty by obtaining the use of an electronic computer.

Finally, we note that the calculated meridional cellular circulation is sufficiently realistic to consider also a solution of the thermodynamic as well as the dynamic equations by constructing, as in this paper, series representations of the velocity components in powers of Ω, each coefficient being a series in powers of ΔT, the difference in mean temperature between equator and pole.

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