Energy Transfer in an Isotropic Turbulent Flow[1]

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Abstract. This paper examines the dynamical consequence of the hypothesis that fourth-order mean values of the fluctuating velocity components are related to second-order mean values as they would be for a normal joint-probability distribution. The equations derived by Tatsumi for isotropic turbulence on the basis of this hypothesis are integrated numerically as an initial value problem for an inviscid fluid. The most remarkable feature revealed by the computation is that the energy spectrum function becomes negative during the course of time in certain regions of wave-number space. This situation is similar to the result obtained previously for two-dimensional turbulence. Truncation errors that arise from finite-difference approximations in numerical integration are examined. It is tentatively concluded that this unphysical negative energy is not generated by the truncation errors but is the consequence of the quasi-normality hypothesis.

1. *Introduction.* The problem of homogeneous turbulence can be stated as follows: given the state of the turbulence generated in a fluid at an initial instant, to predict the state of turbulence as a function of time. Although a considerable amount of work has been done on this problem, a satisfactory solution applicable to the major portion of the time-history of turbulence has not been obtained. The chief difficulty lies in obtaining a determinate set of dynamical equations. From the momentum and continuity equations, equations involving moments (correlation functions) of the fluctuating velocity components of any order can be constructed. Each n-order equation so obtained involves the moments of $n + 1$ order as a direct consequence of the nonlinearity of the Navier-Stokes equations.

The various methods of approximation proposed for making the infinite set of dynamical equations finite can be divided into two broad classes. In the first class, models of dynamical processes are postulated on physical grounds; examples are the theories of Kolmogoroff and Heisenberg. The second class consists of schemes for systematic analytical approximations.

So far two different approaches have been taken in the second class. The most straightforward sequence for closing the infinite set of moment equations consists of ignoring moments of $n + 1$ order in equations for n-order moments [*Deissler,* 1958 and 1960]. It is possible that there may be a fundamental limitation to this scheme. Whereas we may reasonably expect the approximation to converge rapidly when the Reynolds number for the system is small, we do not know whether this scheme yields adequate approximations for interesting cases of the very large Reynolds number.

Another approach is to introduce the hypothesis that the fourth-order cumulants of the velocity field are zero, that is, that fourth-order moments of the distribution of simultaneous velocity components are related to second-order moments as they would be for a normal-probability distribution. Fourth-order moments can be then expressed in terms of second-order moments, and the set of moment equations is closed. *Proudman and Reid* [1954] and *Tatsumi* [1957] applied this hypothesis to the problem of decay of incompressible isotropic turbulence.

In a recent study on the mathematical structure of isotropic turbulence in two dimensions, *Reid* [1959] and *Ogura* [1962, hereafter referred to as paper A] derived a set of dynamical equations. These equations follow from the above hypothesis and are the two-dimensional counterpart of the equations derived by *Proudman and*

[1] Based on a paper presented at the International Symposium on Fundamental Problems in Turbulence and Their Relation to Geophysics sponsored by the International Union of Geodesy and Geophysics and the International Union of Theoretical and Applied Mechanics, held September 4–9, 1961, in Marseilles, France.

Reid [1954] and *Tatsumi* [1957]. Ogura then solved the equations numerically as an initial value problem. The time-history of two-dimensional turbulence so calculated displays a substantially different behavior from that generally accepted for three-dimensional turbulence. The calculated rate of energy transfer is found to be greater toward the larger than toward the smaller scales. Perhaps the most noteworthy feature revealed by the calculation is that the energy spectrum eventually becomes negative for medium-sized eddies. It was concluded in paper A that this unphysical negative energy cannot possibly be generated by the truncation errors associated with the finite-difference approximations in numerical integration but is the consequence of the quasi-normality hypothesis.

Since this hypothesis has been used by several authors, the study has been extended in order to investigate its consequences for three-dimensional turbulence. The purpose of this paper is to report the results so far obtained.

2. *Design of numerical integration.* By changing one of the independent variables from μ to κ'', the set of dynamical equations derived by *Tatsumi* [1957, equations 2.6 and 2.18] are transformed to

$$\frac{\partial}{\partial t} E(\kappa, t) + 2\nu\kappa^2 E(\kappa, t) = \int_0^\infty \int_{|\kappa-\kappa'|}^{\kappa+k'} \Phi(\kappa, \kappa', \kappa'', t)\, d\kappa''\, d\kappa' \quad (2.1)$$

$$\frac{\partial \Phi}{\partial t} + \nu(\kappa^2 + \kappa'^2 + \kappa''^2)\Phi = \varphi_1(\kappa, \kappa', \kappa'')E(\kappa, t)E(\kappa', t) + \varphi_2(\kappa, \kappa', \kappa'')E(\kappa', t)E(\kappa'', t) + \varphi_3(\kappa, \kappa', \kappa'')E(\kappa'', t)E(\kappa, t) \quad (2.2)$$

where E is the energy spectrum function, ν the kinetic viscosity coefficient, κ the wave number,

$$\varphi_1 = \frac{q}{16\kappa^3\kappa'^3\kappa''}(-3\kappa^2\kappa''^2 + 2\kappa^2\kappa'^2 - \kappa'^4 - \kappa^4 + \kappa'^2\kappa''^2)$$

$$\varphi_2 = \frac{q}{16\kappa\kappa'^3\kappa''^3}(3\kappa^2\kappa''^2 - 2\kappa'^2\kappa''^2 + \kappa''^4 - \kappa'^2\kappa^2 + \kappa'^4)$$

$$\varphi_3 = \frac{q}{16\kappa^3\kappa'\kappa''^3}(\kappa^4 - \kappa^2\kappa'^2 - \kappa''^2 + \kappa'^2\kappa''^2)$$

and q is the symmetric quartic

$$q = 2\kappa^2\kappa'^2 + 2\kappa^2\kappa''^2 + 2\kappa'^2\kappa''^2 - \kappa^4 - \kappa'^4 - \kappa''^4$$

The set of (2.1) and (2.2) constitutes the fundamental equations for the present study of turbulence.

In numerical integration of this set of equations, the infinite integration in (2.1) is necessarily truncated at a finite limit, say κ^*. It can then be shown that the total energy is still conserved, for an inviscid fluid, in the form

$$\frac{\partial}{\partial t}\int_0^{\kappa^*} E(\kappa, t)\, d\kappa = 0 \quad (2.3)$$

if the upper and lower limits of the integration with respect to κ'' are also truncated at κ^*.

For convenience of numerical analysis, dimensionless variables are introduced in the following form:

$$t = \tau(\Delta t) \qquad \tau = 0, 1, 2, \cdots$$

$$\kappa = k(\Delta\kappa) \qquad \kappa' = j(\Delta\kappa) \qquad \kappa'' = i(\Delta\kappa)$$

$$\kappa^* = I(\Delta\kappa) \qquad i, j, k = 0, 1, 2, \cdots, I$$

$$E = E_0\hat{E}(k, \tau)$$

$$\Phi = \Phi_0\psi(k, j, i, \tau) \quad (2.4)$$

where E_0 and Φ_0 are constants. Δt and $\Delta\kappa$ denote the finite-difference increments for time and wave number, respectively. Equations 2.1 and 2.2 then take the following dimensionless finite-difference form:

$$\hat{E}^{(\tau+1)}(k) = \left[1 - \frac{k^2}{R_e}\right]\left[1 + \frac{k^2}{R_e}\right]^{-1}\hat{E}^{(\tau)}(k) + \sigma_1\left[1 + \frac{k^2}{R_e}\right]^{-1} \cdot \int_0^I \int_{|k-j|\le I}^{k+j\le I} \psi^{(\tau+1/2)}(k, j, i, \tau)\, di\, dj \quad (2.4)$$

$$\psi^{(\tau+1/2)} = \left[1 - \frac{1}{2R_e}(k^2 + j^2 + i^2)\right] \cdot \left[1 + \frac{1}{2R_e}(k^2 + j^2 + i^2)\right]^{-1}\psi^{(\tau-1/2)} + \sigma_2\left[1 + \frac{1}{2R_e}(k^2 + j^2 + i^2)\right]^{-1} \cdot [\hat{\varphi}_1\hat{E}^{(\tau)}(k)\hat{E}^{(\tau)}(j) + \hat{\varphi}_2\hat{E}^{(\tau)}(j)\hat{E}^{(\tau)}(i) + \hat{\varphi}_3\hat{E}^{(\tau)}(i)\hat{E}^{(\tau)}(k)] \quad (2.5)$$

where

$$R_e = \nu^{-1}(\Delta t)^{-1}(\Delta \kappa)^{-2}$$

$$\sigma_1 = (\Delta t)(\Delta \kappa)^2 \Phi_0 E_0^{-1}$$

$$\sigma_2 = (\Delta t)(\Delta \kappa) E_0^2 \Phi_0^{-1}$$

and the dimensionless form of φ_1, φ_2, and φ_3 are obtained simply by replacing κ, κ', and κ'' by k, j, and i, respectively. In these equations, a typical term like $\hat{E}^{(\tau)}(k)$ represents a value of $\hat{E}$ at $t = \tau(\Delta\tau)$ and $\kappa = k(\Delta\kappa)$. In deriving (2.4) and (2.5), a viscous term like νE in (2.1) has been replaced by $\nu(E^{(\tau+1)} + E^{(\tau)})/2$. This finite-difference form reduces the truncation error and also permits us to use a larger Δt, without violating computational stability, than is permitted using a simple form like $\nu\hat{E}^{(\tau)}$ (see Appendix).

To integrate (2.4) and (2.5) it is necessary to assume initial conditions for $\hat{E}$ and ψ. As an initial spectral distribution we take the following form:

$$\hat{E}^{(0)}(k) = \frac{8}{3\sqrt{\pi}\, k_0}\left(\frac{k}{k_0}\right)^4 \exp\left[-\left(\frac{k}{k_0}\right)^2\right] \quad (2.6)$$

where $k_0 = \kappa_0/\Delta\kappa$ and κ_0 is a constant. The form of (2.6) has also been used by *Proudman and Reid* [1954].

The initial distribution of ψ remains to be specified. In view of the fact that no experimental information is available, we shall assume that the initial energy transfer between eddies of different sizes is zero; that is,

$$\psi^{(1/2)}(k, j, i) = 0 \quad (2.7)$$

The choice of this condition is made on the basis of simplicity.

3. *Result of numerical integration.* Two different integrations of (2.4) and (2.5) have been completed for an inviscid fluid ($R_e \rightarrow \infty$) on an IBM 709 computer, starting from the identical initial conditions (2.6) and (2.7). The only difference between the two runs is that I, the upper limit of the integration, is 16 in the first run and 32 in the second. The following values are assigned to the dimensionless parameters:

$$k_0 = 4 \qquad \sigma_1 = 0.00625 \qquad \sigma_2 = 0.025$$

Figures 1 and 2 show the dimensionless energy spectrum function $\hat{E}$ plotted against the dimensionless wave number k for various values of time. Figure 3 shows the energy transfer function as a function of time and wave number. The transfer function gives the net energy transfer into a wave-number band from all other wave numbers. We observe in Figures 1 to 3 that, in marked contrast with the case for turbulence in two dimensions, the energy is transferred persistently from larger to smaller eddies, except for the very small 'back transfer' of energy to the region of very small wave numbers.

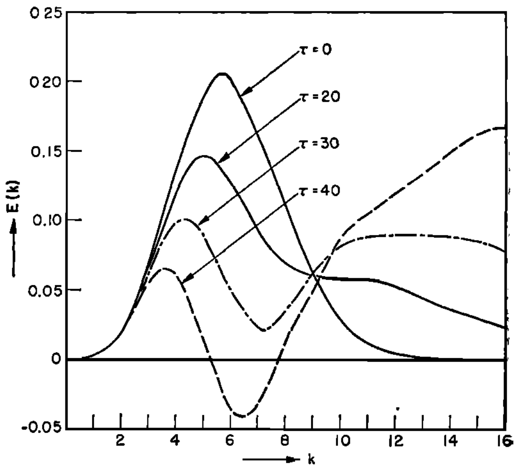

Fig. 1. Variation of energy spectrum with time. $I = 16$.

The most noteworthy feature of the computation is that negative energy appears again during the course of time in the region of energy-containing eddies. In paper A, the truncation errors induced by the finite-difference approximations have been examined in detail. It has been concluded that the truncation error that arises from replacing derivatives and integrations by finite differences and summations respectively is very small for the Δt and $\Delta\kappa$ used in that calculation. The dynamical equations for turbulence in three dimensions are similar in mathematical form to those in two dimensions. The space increment used in this paper is the same as that in paper A. Consequently we may reasonably assume that the generation of negative energy is not caused by this type of truncation error. This may also be justified by the fact that the total energy computed from the numerical solutions remains approximately constant, as is required in equation 2.3 (see Table 1). Perhaps the most serious error is that induced by truncating the domain of integration at the

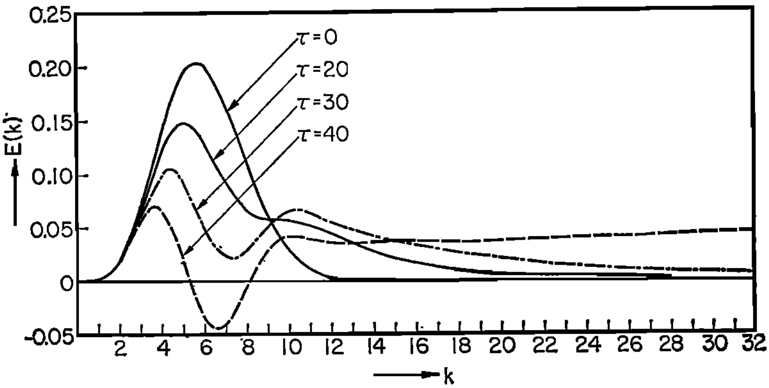

Fig. 2. The same as Figure 1 except $I = 32$.

finite wave number I. It is serious because, as we observe in Figures 1 to 3, this truncating process is equivalent to preventing energy transfer past the limiting wave number. The piling-up of energy appearing near and at the end of wave-number space is apparently caused by this effect. The error from this source is greater in three than in two dimensions, because only a small fraction of energy is transferred to the region of large wave numbers in two dimensions. It was found in paper A that $I = 32$ is large enough to make the error from this source sufficiently small; in three dimensions, it apparently is not.

This situation may be demonstrated further by examining the production of vorticity, because the total vorticity is more sensitive than the total energy to the error involved in the region of large wave numbers. It was pointed out by *Proudman and Reid* [1954] that, when $\nu = 0$, (2.1) and (2.2) permit the following simple equation (as far as an inviscid fluid is concerned, the set of equations derived by Tatsumi is identical with the result given by Proudman and Reid):

$$\frac{d^2}{dt^2}\int_0^\infty \kappa^2 E(\kappa)\,d\kappa = \frac{2}{3}\left[\int_0^\infty \kappa^2 E(\kappa)\,d\kappa\right]^2 \quad (3.1)$$

The equation for the mean square value of one component of vorticity may then be written in dimensionless form

$$\frac{d^2\bar{\omega}^2}{dt^2} = \gamma(\bar{\omega}^2)^2 \quad (3.2)$$

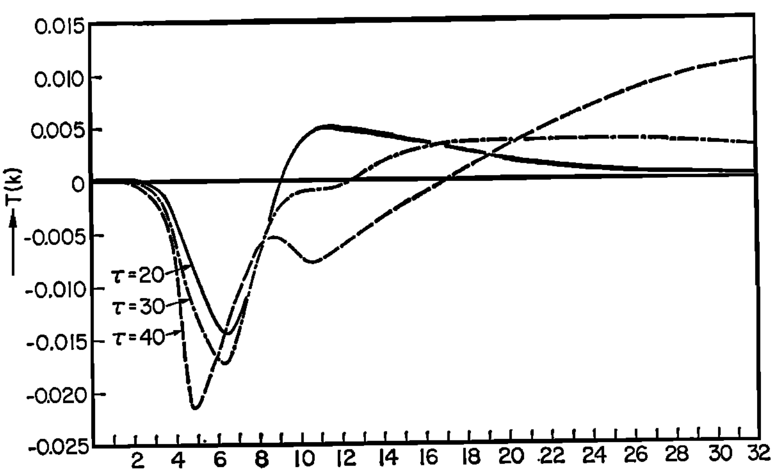

Fig. 3. Variation of energy transfer function with time.

TABLE 1. Total Energy as a Function of Time in Percentage of Its Initial Value

Time	Total Energy
0	100.0000
10	99.9994
20	99.9980
30	99.9980
40	100.0029

where

$$\bar{\omega}^2 = \frac{2}{3}\int_0^\infty k^2\hat{E}(k)\,dk$$

and

$$\gamma = E_0(\Delta t)^2(\Delta\kappa)^3$$

The τ and k in (3.2) are dimensionless, continuous variables, scaled by Δt and $\Delta\kappa$, respectively. The solution of (3.2) is the Weierstrassian elliptic function

$$\bar{\omega}^2 = 2^{2/3}\bar{\omega}_0^{\,2}\wp(t';\,0,\,1) \qquad (3.3)$$

where $\bar{\omega}_0^2$ is a constant corresponding to the initial value of $\bar{\omega}^2$, and, with a suitable choice of the origin of time,

$$t' = 2^{1/3}(\tfrac{1}{6}\gamma\bar{\omega}_0^{\,2})^{1/2}\tau$$

For the initial distribution of energy spectrum (2.6), $\bar{\omega}_0^2$ is given by

$$\bar{\omega}_0^{\,2} = 5/3(\kappa_0/\Delta\kappa)^2$$

Consequently

$$t' = 2^{1/3}\left[\frac{5}{18}\,E_0(\Delta t)^2(\Delta\kappa)^3(\Delta\kappa/\kappa_0)\right]^{1/2}\tau$$

$$= 0.03320\tau \qquad (3.4)$$

In the physical problem, only one real period of the doubly periodic elliptic function is relevant, namely, $0 \le t' \le 2T$, where $T = 1.53$. The relation 3.4 indicates then that $T = 1.53$ corresponds roughly to $\tau = 46$.

Figure 4 compares $\bar{\omega}^2/\bar{\omega}_0^2$ calculated from the exact solution (3.3) with those calculated from the numerical solution for the case of $I = 32$. We observe that the numerically calculated result is initially in good agreement with the exact solution, but as time increases a substantial fraction of energy reaches the end of wave-number space and thereby the difference between the two results becomes large. It should be emphasized, however, that the variation of the spectrum function with time in the middle range of space appears to be little influenced by the variation taking place at and near the end of wave-number space. This can be seen in Figure 5, which compares the variation of spectrum with time at $k = 7$, where the first negative energy appears, for the different I. We observe that the difference between the two cases is surprisingly small compared with the total change that $\hat{E}(k = 7)$ undergoes during the whole period of time.

4. *Concluding remarks.* As was described in paper A, the hypothesis of zero-fourth-order cumulants has been used by several authors in their investigations of the mathematical structure of turbulence. Some experimental results have also been reported that seemed to support the validity of this hypothesis. But also some work has cast doubt on the applicability of this hypothesis to turbulence problems. As was mentioned in section 3, *Proudman and Reid* [1954] have been able to integrate the equation for the production of vorticity exactly and thereby deduce the value of the skewness factor. The resulting skewness factor takes the values which, according to inequalities derived by *Betchov* [1956], are incompatible with a positive-definite distribution having zero-fourth-order cumulants. *Kraichnan* [1961] has demonstrated analyti-

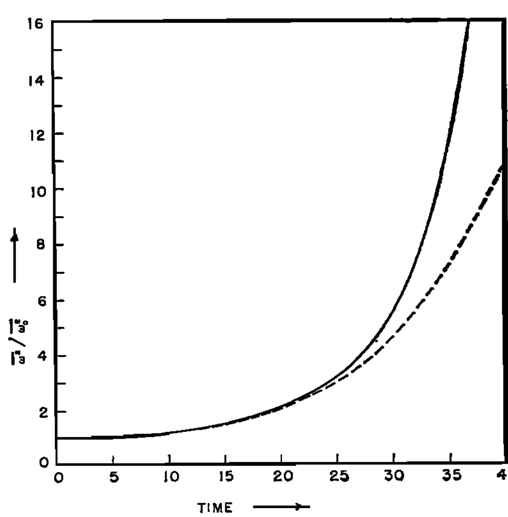

Fig. 4. Comparison of production of vorticity calculated from the exact solution (solid line) and the numerical solution (dashed line).

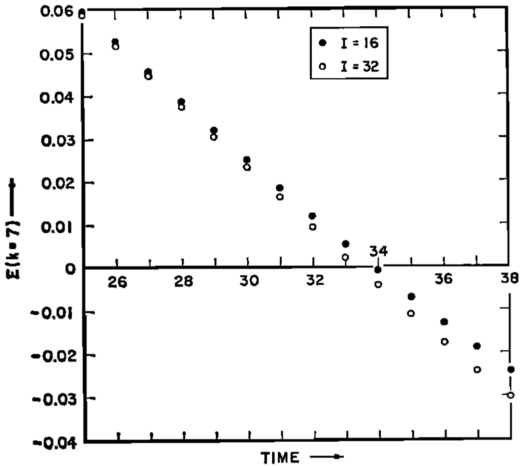

Fig. 5. Comparison of variation of energy spectrum with time at $k = 7$ for $I = 16$ and $I = 32$.

cally that this hypothesis leads to negative-definite power spectrum when it is applied to the 'convection' of a *scalar* field by a prescribed random velocity field. He has also suggested the possibility of having a nonpositive spectrum in some regions of wave-number space when it is applied to a *vector* field.

The results of the calculations reported in this paper for a three-dimensional velocity field show a generation of negative energy, similar to that reported in paper A. As described in section 3, the numerical error induced by truncating the domain of integration at a finite wave number is more serious for turbulence in three dimensions than for turbulence in two dimensions. Consequently, the conclusion that the generation of negative energy is most likely the consequence of the hypothesis so applied should be regarded as tentative. A more definite conclusion may be reached when the viscous forces are incorporated with the inertia forces in the calculation, so that no appreciable amount of energy appears at the end of truncated wave-number space. This calculation is under way, and results will be reported soon.

Appendix

The computational stability of the finite-difference equations (2.4) and (2.5) is difficult to investigate because the equations are nonlinear. As for the effect of viscous terms on the stability conditions, considerable information can be obtained by examining the computational stability of simplified versions of the basic equations. The equations we treat for this purpose are

$$\partial E/\partial t = a\psi - \nu E \tag{A.1}$$

$$\partial \psi/\partial t = bE - \nu\psi \tag{A.2}$$

where a, b, and ν are constants ($\nu \geq 0$, $ab < 0$).

We write (A.1) and (A.2) in finite-difference form:

$$\left.\begin{aligned} E^{(\tau+1)} - E^{(\tau)} &= a(\Delta t)\psi^{(\tau+1/2)} \\ &\qquad - \sigma(E^{(\tau+1)} + E^{(\tau)}) \\ \psi^{(\tau+1/2)} - \psi^{(\tau-1/2)} &= b(\Delta t)E^{(\tau)} \\ &\qquad - \sigma(\psi^{(\tau+1/2)} + \psi^{(\tau-1/2)}) \end{aligned}\right\} \tag{A.3}$$

where $\sigma = \nu(\Delta t)/2$ and dimensionless time τ is taken as an integer.

Equations A.3 may be written in matrix form:

$$\begin{bmatrix} E^{(\tau+1)} \\ \psi^{(\tau+1/2)} \end{bmatrix} = \tilde{G}(\Delta t) \begin{bmatrix} E^{(\tau)} \\ \psi^{(\tau-1/2)} \end{bmatrix}$$

where

$$\tilde{G}(\Delta t) = \begin{bmatrix} \dfrac{1-\sigma}{1+\sigma} + \dfrac{ab(\Delta t)^2}{(1+\sigma)^2} & \dfrac{a(\Delta t)}{1+\sigma}\left(\dfrac{1-\sigma}{1+\sigma}\right) \\ \dfrac{b(\Delta t)}{1+\sigma} & \dfrac{1-\sigma}{1+\sigma} \end{bmatrix}$$

is what we call an amplification matrix. The finite difference equations A.3 are then computationally stable if the absolute values of the eigenvalues of $\tilde{G}$ are equal to or less than 1 [*Richtmyer*, 1957].

The eigenvalues of $\tilde{G}$ are roots of the quadratic equation

$$(1+\sigma)^2\omega^2 - [2(1-\sigma^2) - \gamma^2]\omega + (1-\sigma)^2 = 0$$

where $\gamma^2 = -ab(\Delta t)^2 > 0$.

First, we can readily show that, when $\nu = 0$ (that is, $\sigma = 0$), the stability condition ($|\omega| \leq 1$) is met if $\gamma^2 \leq 4$. When $\sigma \neq 0$, it is found after some manipulation that the condition is met if

$$\gamma^2 + \gamma[\gamma^2 + 4(\sigma^2 - 1)]^{1/2} \leq 4(1+\sigma)$$

It is obvious that this condition further reduces to the simple one

$$\gamma^2 \leq 4 \qquad \text{(A.4)}$$

This result indicates that, as far as γ^2 satisfies (A.4), the finite-difference equations A.3 are always stable irrespective of values of σ. This would not follow if the viscous terms νE and $\nu\psi$ are evaluated as $\nu E^{(\tau)}$ and $\nu\psi^{(\tau+1/2)}$.

Note added in proof. The calculation has been completed for several values of the Reynolds number. The generation of negative energy is also observed in those cases where the Reynolds number is high and yet no appreciable amount of energy appears at the end of the truncated wave-number space.

Acknowledgments. This research was sponsored by the National Science Foundation under contract G18985. I wish to thank Mrs. Masako Ogura for programming the computation, part of which was performed at the MIT Computation Center, Cambridge, Massachusetts. Thanks are also due to Mr. and Mrs. William Blumen for their help in preparing the manuscript.

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