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Perturbation Analysis of the Navier-Stokes Equations in Lagrangian Form with Selected Linear Solutions[1]

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Abstract. The Navier-Stokes equations for incompressible flow in their Lagrangian form are taken as a starting point. A perturbation technique is then used to obtain first- and second-order sets of equations, and the general procedure for solving the equations to any order is given. The first-order equations yield interesting two- and three-dimensional motions that have some of the properties of 'stirring,' 'eddies,' and 'turbulence'; it is suggested that various problems in turbulent motion might possibly be re-examined by means of these equations.

Introduction. The Navier-Stokes equations in their Eulerian form have proved to be so intractable that nearly all hope of finding a meaningful solution to describe even one possible fluid motion to within its finest small-scale structure has been abandoned. Present-day thinking is along the line of finding meaningful averages over an ensemble of possible solutions so as to extract the part of the fluid motion that is believed to be repeatable from one observation to the next.

However, the Navier-Stokes equations can also be written in a Lagrangian form, as was pointed out by *Gerber* [1949] and discussed by *Corrsin* [1961*b*]. In Lagrangian form, these equations, even for incompressible flow, are complicated in appearance. The complexity arises in ways that are quite different from the source of the complexity in the Eulerian form, so that, when a fluid parcel is followed, the nonlinear field acceleration terms of the Eulerian form vanish. This one feature of the perturbation expansion of the Lagrangian equations makes it possible to obtain motions that appear to have features much more like turbulence than those possible from studies of the linearized Eulerian equations.

The present status of knowledge about turbulent flow has been described by *Corrsin* [1959, 1961*a*, *b*]. *Stewart* [1959] has given a description of the natural occurrence of turbulence applicable to geophysical problems and put forth the suggestion that 'a fluid is said to be turbulent if each component of the vorticity is distributed irregularly and aperiodically in time and space, if the flow is characterized by a transfer of energy from larger to smaller scales of motion, and if the mean separation of neighboring fluid particles tends to increase with time.'

The practical and the theoretical solutions to the problem of turbulence are still far apart. As summarized by *Kraichnan* [1961], the theoretical solutions serve only as a check against one another. The practical solutions still seem to have many deficiencies.

Similar to the problems of turbulence are the problems of stirring, mixing, and dispersion [*Eckart*, 1948; *Corrsin*, 1961*a*]. Here Lagrangian solutions seem advantageous in explaining how very strong small-scale gradients can be produced by stirring so as to augment many fold the effect of mixing and molecular dispersion. Similarly, if a particle is to be followed, Lagrangian equations are the natural way to seek knowledge of its motion.

In meteorology and oceanography, 'eddy' viscosity, 'eddy' conduction, 'eddy diffusion, mixing length,' and 'austausch' coefficients have replaced the more fundamental concepts of molecular viscosity, molecular diffusion, and molecular conduction, solely because the original equations in terms of the latter quantities have simply not yielded solutions capable of explain-

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ing the fluxes that are observed to occur. Yet, in the true physical sense, the equations involving the molecular concepts are more meaningful than those involving 'eddy' concepts.

Fine-structure gradients of velocity and temperature are known to be large, and, if solutions could be found for the 'molecular' equations that could explain the 'gross' features as observed, our understanding of these features would be greatly increased. The 'eddies' would be a part of the solution instead of something averaged out of the solution by an averaging process that unfortunately changes the nature of the 'eddies' and the mean flow as its time and spatial scale is changed.

If meaningful solutions to the 'molecular' equations could be found, it might even be possible to fill in some of the gaps in the classical literature on turbulence. For example, Reynolds' classical experiment with filaments of dye in a pipe *describes but does not explain* the onset of turbulent flow at a certain Reynolds number. Also, much of the present work on turbulence is more of an attempt to describe it than to explain it.

Solutions to the Lagrangian equations are difficult both to interpret and to verify, as verification is usually possible only by means of measurements that are more suitably made with reference to a fixed point of observation. In principle, however, a solution in Lagrangian form can be inverted to obtain an approximation to a solution in Eulerian form. Such a solution need not be a solution of the Eulerian equations to any particular order. In fact, a linear solution in Lagrangian form, for at least one nonviscous problem that has been studied, provided a reasonably exact facsimile to a third-order solution of the same problem in Eulerian form.

The problem of linearization. If the Navier-Stokes equations in their Eulerian form are linearized, the result, in the absence of field accelerations, is a set of equations from which it appears that only trivial solutions quite unlike turbulent flow can be obtained. On the contrary, if the equations are transformed to their Lagrangian form, the linear and second-order equations appear to preserve certain realistic features of turbulent flow. In the Lagrangian form, the acceleration following a particle is preserved in the linear equations. A linearization of the Lagrangian equations in the study of gravity waves, for example, has yielded much more realistic waves than the linearized Eulerian equations [*Miche,* 1944; *Pierson,* 1961].

The Lagrangian equations in their nonviscous form have not been studied as extensively as the Eulerian equations. An exception is a paper by *Eckart* [1960], in which many important properties of these equations were established. Unpublished work of Eckart suggests that the application of classical perturbation procedures may yield interesting results when applied to the Lagrangian equations.

Certain points raised by *Corrsin* [1961*a*] are also of interest here. Although the complete equations are 'even more severely nonlinear' than the Eulerian equations, the perturbation analysis appears to yield a system of linear equations such that an approximate solution of the linear equations may be a good start toward a correct solution of a problem in fluid motion, and, as was pointed out above, the nonlinearities appear in a different way so as to preserve certain desirable features even in the linear equations.

The equation of continuity. The equation of continuity for incompressible flow in Lagrangian form is given by *Lamb* [1932] as

$$\frac{\partial(x, y, z)}{\partial(\alpha, \beta, \delta)} = 1 \tag{1}$$

In this work, the tags for the fluid particles will be identified with their coordinates either at zero time or in the undisturbed position. This greatly simplifies the perturbation form of the equation of continuity. (The a, b, and c used in Lamb will be replaced by α, β, and δ throughout.)

The other possible form of the equation of continuity, namely, that

$$\frac{\partial(x_o, y_o, z_o)}{\partial(\alpha, \beta, \delta)} = \frac{\partial(x, y, z)}{\partial(\alpha, \beta, \delta)} \tag{2}$$

introduces a complication due to the fact that the tags α, β, and δ no longer refer to the initial coordinates of the fluid parcels.

Derivation of the Lagrangian equation. For incompressible flow, the Navier-Stokes equations are given by equations 3, where subscripts denote partial differentiation [*Lamb,* 1932].

$$x_{tt} - (\mu/\rho)\nabla^2 u = -p_x/\rho \quad (3)$$
$$y_{tt} - (\mu/\rho)\nabla^2 v = -p_y/\rho$$
$$z_{tt} + g - (\mu/\rho)\nabla^2 w = -p_z/\rho$$

In equations 3, $x_{tt} = u_t + uu_x + vu_y + wu_z$ and $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. If the first equation, following *Lamb* [1932], is multiplied by x_α, the second by y_α, and the third by z_α, and if then the three equations are added, the result is the first of three hybrid equations that can be obtained. Note that $p_x x_\alpha + p_y y_\alpha + p_z z_\alpha$ is equal to p_α.

$$\left(x_{tt} - \frac{\mu}{\rho}\nabla^2 u\right)x_\alpha + \left(y_{tt} - \frac{\mu}{\rho}\nabla^2 v\right)y_\alpha + \left(z_{tt} + g - \frac{\mu}{\rho}\nabla^2 w\right)z_\alpha = \frac{-p_\alpha}{\rho} \quad (4)$$

Two similar equations that involve x_β, y_β, and z_β, and x_δ, y_δ, and z_δ can also be obtained. From equation 1, the determinant of these equations is 1 if the terms in parentheses are solved for. Thus the equations for x_{tt}, y_{tt}, and z_{tt} are given by

$$x_{tt} = -\frac{1}{\rho}\frac{\partial(p, y, z)}{\partial(\alpha, \beta, \delta)} + \frac{\mu}{\rho}\nabla^2 u \quad (5)$$
$$y_{tt} = -\frac{1}{\rho}\frac{\partial(x, p, z)}{\partial(\alpha, \beta, \delta)} + \frac{\mu}{\rho}\nabla^2 v$$
$$z_{tt} + g = -\frac{1}{\rho}\frac{\partial(x, y, p)}{\partial(\alpha, \beta, \delta)} + \frac{\mu}{\rho}\nabla^2 w$$

The only remaining difficulty is expressing the ∇^2 operator in Lagrangian form. Following *Gerber* [1949] and *Corrsin* [1961*b*], we can write $\nabla^2 u$ in Lagrangian form as in equation 6.

$$\nabla^2 u = \frac{\partial\left(\dfrac{\partial(x_t, y, z)}{\partial(\alpha, \beta, \delta)}, y, z\right)}{\partial(\alpha, \beta, \delta)} + \frac{\partial\left(x, \dfrac{\partial(x, x_t, z)}{\partial(\alpha, \beta, \delta)}, z\right)}{\partial(\alpha, \beta, \delta)} + \frac{\partial\left(x, y, \dfrac{\partial(x, y, x_t)}{\partial(\alpha, \beta, \delta)}\right)}{\partial(\alpha, \beta, \delta)} \quad (6)$$

Similar expressions would result for $\nabla^2 v$ and $\nabla^2 w$.

The first- and second-order perturbation equations. The full nonlinear form of these equations in Lagrangian form appears even more formidable than the original Eulerian form. Assume a perturbation about $x = \alpha$, $y = \beta$, $z = \delta$, and $p = p_0 - g\rho\delta$ of the form

$$x = \alpha + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots \quad (7)$$
$$y = \beta + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots$$
$$z = \delta + \epsilon z_1 + \epsilon^2 z_2 + \epsilon^3 z_3 + \cdots$$
$$p = p_0 - g\rho\,\delta + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \cdots$$

In (7), ϵ can be considered to be an ordering parameter that can be set equal to 1 in expressing a final solution. When (7) is substituted into all appropriate equations above and when terms in equal powers of ϵ are collected and set equal to each other, the zero-order terms balance out exactly. The first-order terms are given by equations 8, where $\nabla_L^2 x_{1t} = x_{1t\alpha\alpha} + x_{1t\beta\beta} + x_{1t\delta\delta}$.

$$x_{1tt} + gz_{1\alpha} + \frac{p_{1\alpha}}{\rho} - \frac{\mu}{\rho}\nabla_L^2 x_{1t} = 0 \quad (8)$$
$$y_{1tt} + gz_{1\beta} + \frac{p_{1\beta}}{\rho} - \frac{\mu}{\rho}\nabla_L^2 y_{1t} = 0$$
$$z_{1tt} + gz_{1\delta} + \frac{p_{1\delta}}{\rho} - \frac{\mu}{\rho}\nabla_L^2 z_{1t} = 0$$
$$x_{1\alpha} + y_{1\beta} + z_{1\delta} = 0$$

Just as in many perturbation schemes, a solution to (8) can be used to obtain the right-hand side of the second-order equations, and the second-order solution can also in principle be found. The second-order equations are given either by equations 9 or by equations 10.

$$x_{2tt} + \frac{p_{2\alpha}}{\rho} + gz_{2\alpha} - \frac{\mu}{\rho}\nabla_L^2 x_{2t} = \frac{p_{1\alpha}x_{1\alpha}}{\rho} \quad (9)$$
$$+ \frac{p_{1\beta}y_{1\alpha}}{\rho} + \frac{p_{1\delta}z_{1\alpha}}{\rho} - g(z_{1\alpha}y_{1\beta} - y_{1\alpha}z_{1\beta})$$
$$- \frac{2\mu}{\rho}(x_{1\alpha}x_{1t\alpha\alpha} + y_{1\beta}x_{1t\beta\beta} + z_{1\delta}x_{1t\delta\delta})$$
$$- \frac{2\mu}{\rho}((y_{1\alpha} + x_{1\beta})x_{1t\beta\alpha} + (z_{1\alpha} + x_{1\delta})x_{1t\alpha\delta}$$
$$+ (z_{1\beta} + y_{1\delta})x_{1t\beta\delta}) - \frac{\mu}{\rho}(x_{1t\alpha}\nabla_L^2 x_1$$
$$+ x_{1t\beta}\nabla_L^2 y_1 + x_{1t\delta}\nabla_L^2 z_1)$$

$$y_{2tt} + \frac{p_{2\beta}}{\rho} + gz_{2\beta} - \frac{\mu}{\rho}\nabla_L^2 y_{2t} = \frac{p_{1\alpha}x_{1\beta}}{\rho}$$
$$+ \frac{p_{1\beta}y_{1\beta}}{\rho} + \frac{p_{1\delta}z_{1\beta}}{\rho} - g(x_{1\beta}z_{1\alpha} - x_{1\alpha}z_{1\beta})$$
$$- \frac{2\mu}{\rho}(x_{1\alpha}y_{1t\alpha\alpha} + y_{1\beta}y_{1t\beta\beta} + z_{1\delta}y_{1t\delta\delta})$$
$$- \frac{2\mu}{\rho}((y_{1\alpha} + x_{1\beta})y_{1t\alpha\beta} + (z_{1\alpha} + x_{1\delta})y_{1t\alpha\delta}$$
$$+ (z_{1\beta} + y_{1\delta})y_{1t\beta\delta}) - \frac{\mu}{\rho}(y_{1t\alpha}\nabla_L^2 x_1$$
$$+ y_{1t\beta}\nabla_L^2 y_1 + y_{1t\delta}\nabla_L^2 z_1)$$

$$z_{2tt} + \frac{p_{2\delta}}{\rho} + gz_{2\delta} - \frac{\mu}{\rho}\nabla_L^2 z_{2t} = \frac{p_{1\alpha}x_{1\delta}}{\rho}$$
$$+ \frac{p_{1\beta}y_{1\delta}}{\rho} + \frac{p_{1\delta}z_{1\delta}}{\rho} - g(x_{1\alpha}y_{1\beta} - x_{1\beta}y_{1\alpha})$$
$$- \frac{2\mu}{\rho}(x_{1\alpha}z_{1t\alpha\alpha} + y_{1\beta}z_{1t\beta\beta} + y_{1\delta}z_{1t\delta\delta})$$
$$- \frac{2\mu}{\rho}((y_{1\alpha} + x_{1\beta})z_{1t\alpha\beta} + (z_{1\alpha} + x_{1\delta})z_{1t\alpha\delta}$$
$$+ (z_{1\beta} + y_{1\delta})z_{1t\beta\delta}) - \frac{\mu}{\rho}(z_{1t\alpha}\nabla_L^2 x_1$$
$$+ z_{1t\beta}\nabla_L^2 y_1 + z_{1t\delta}\nabla_L^2 z_1)$$

(plus the equation of continuity as in 10).

$$x_{2tt} + gz_{2\alpha} + \frac{p_{2\alpha}}{\rho} - \frac{\mu}{\rho}\nabla_L^2 x_{2t} \tag{10}$$
$$= -(x_{1tt}x_{1\alpha} + y_{1tt}y_{1\alpha} + z_{1tt}z_{1\alpha})$$
$$+ \frac{\mu}{\rho}[(x_{1\alpha}\nabla_L^2 x_{1t} + y_{1\alpha}\nabla_L^2 y_{1t}$$
$$+ z_{1\alpha}\nabla_L^2 z_{1t}) - 2(x_{1t\alpha\alpha}(x_{1\alpha}) + x_{1t\beta\beta}(y_{1\beta})$$
$$+ x_{1t\delta\delta}(z_{1\delta}) + x_{1t\alpha\beta}(y_{1\alpha} + x_{1\beta})$$
$$+ x_{1t\alpha\delta}(z_{1\alpha} + x_{1\delta}) + x_{1t\beta\delta}(y_{1\delta} + z_{1\beta}))$$
$$- x_{1t\alpha}\nabla_L^2 x_1 - x_{1t\beta}\nabla_L^2 y_1 - x_{1t\delta}\nabla_L^2 z_1]$$

$$y_{2tt} + gz_{2\beta} + \frac{p_{2\beta}}{\rho} - \frac{\mu}{\rho}\nabla_L^2 y_{2t}$$
$$= -(x_{1tt}x_{1\beta} + y_{1tt}x_{1\beta} + z_{1tt}z_{1\beta})$$
$$+ \frac{\mu}{\rho}[(x_{1\beta}\nabla_L^2 + y_{1\beta}\nabla_L^2 y_{1t} + z_{1\beta}\nabla_L^2 z_{1t})$$
$$- 2(y_{1t\alpha\alpha}(x_{1\alpha}) + y_{1t\beta\beta}(y_{1\beta})$$
$$+ y_{1t\delta\delta}(z_{1\delta}) + y_{1t\alpha\beta}(y_{1\alpha} + x_{1\beta})$$
$$+ (x_{1t\alpha\delta}(z_{1\alpha} + x_{1\delta}) + y_{1t\beta\delta}(y_{1\delta} + z_{1\beta}))$$
$$- y_{1t\alpha}\nabla_L^2 x_1 - y_{1t\beta}\nabla_L^2 y_1 - y_{1t\delta}\nabla_L^2 z_1]$$

$$z_{2tt} + gz_{2\delta} + \frac{p_{2\delta}}{\rho} - \frac{\mu}{\rho}\nabla_L^2 z_{2t}$$
$$= -(x_{1tt}x_{1\delta} + y_{1tt}y_{1\delta} + z_{1tt}z_{1\delta})$$
$$+ \frac{\mu}{\rho}[(x_{1\delta}\nabla_L^2 x_{1t} + y_{1\delta}\nabla_L^2 y_{1t}$$
$$+ z_{1\delta}\nabla_L^2 z_{1t}) - 2(z_{1t\alpha\alpha}(x_{1\alpha}) + z_{1t\beta\beta}(y_{1\beta})$$
$$+ z_{1t\delta\delta}(z_{1\delta}) + z_{1t\alpha\beta}(y_{1\alpha} + x_{1\beta})$$
$$+ z_{1t\alpha\delta}(z_{1\alpha} + x_{1\delta}) + z_{1t\beta\delta}(y_{1\delta} + z_{1\beta}))$$
$$- z_{1t\alpha}\nabla_L^2 x_1 - z_{1t\beta}\nabla_L^2 y_1 - z_{1t\delta}\nabla_L^2 z_1]$$

$$x_{2\alpha} + y_{2\beta} + z_{2\delta} + x_{1\alpha}y_{1\beta} + x_{1\alpha}z_{1\delta}$$
$$+ y_{1\beta}z_{1\delta} - x_{1\delta}z_{1\alpha} - y_{1\delta}z_{1\beta} - x_{1\delta}y_{1\alpha} = 0$$

Equations 9 can be transformed to equations 10 by means of solving for the pressure in equations 8. Except for the correction of some mistakes in algebra, equations 10 are the same as those obtained by a different derivation of the Navier-Stokes equations in Lagrangian form as given in an earlier version of this paper.

Equations 9 and 10 are linear, and their solution is possible on the basis of the standard procedures of linear theories for the solution of partial differential equations. The right-hand side of each equation is, of course, a known function of α, β, δ, and t, once an appropriate solution of equations 8 has been found. Moreover, the perturbation procedure can, in principle, be carried out to any desired order, and the system of equations to be solved is always closed.

Convergence problems. If the perturbation scheme outlined above were carried out to a great many terms, either the solution so obtained would converge more and more closely to an exact solution to the original nonlinear equations or it would, in a sense, blow up with a suggestion that perhaps either the original linear solution was unrealistic or the perturbation expansion was incorrect. There is no a priori reason, however, to expect that the second-order correction will be small for a reasonably strong flow. The series expansion of e^x is valid for all x, and for $x = 2$ the second-order term is as important as the first-order term. Similarly, the second-order correction may be large in such an analysis, and the third or fourth may then turn out to be small.

Three solutions to the linearized equations. Equations 8 appear to be rich enough to yield rather interesting solutions for various kinds of flow subject to the rather strange way in which the equation of continuity fails to be satisfied because of the way that the equations have been linearized. We can imagine that solutions to these equations might provide an understanding of many problems such as the eddying structure of smoke from a smoke stack and the spreading of a patch of dye. Three solutions to this system that represent the spreading of a patch of dye in two-dimensional flow, some strange form of decoupled surface gravity wave, and decaying spatially homogeneous flow that satisfies two of the requirements set forth by Stewart for turbulence have been found; they will be derived and discussed below.

A patch of dye. Under the assumptions that there is no vertical motion, that the motion is the same for all particles with the same δ, and that the motion decays with time, equations 8 can be written as (11)

$$x_{1tt} - \frac{\mu}{\rho}(x_{1t\alpha\alpha} + x_{1t\beta\beta}) = 0 \qquad (11)$$

$$y_{1tt} - \frac{\mu}{\rho}(y_{1t\alpha\alpha} + y_{1t\beta\beta}) = 0$$

$$x_{1\alpha} + y_{1\beta} = 0$$

Subject to the conditions that $x = \alpha$, $y = \beta$ at $t = 0$, and that $x_t = u(x, y) = u(\alpha, \beta)$ and $y_t = v(\alpha, \beta)$ at $t = 0$, the solution is given by equations 12, where V represents a slow drift in the positive x direction. (Note that a low-wave-number term could accomplish the same effect for any desired area in the x, y plane.)

At $t = 0$, $u = x_t|_{t=0}$ and $v = y_t|_{t=0}$ are given by

$$u|_{t=0} = x_t = V + \sum_{pq}\sum a_{pq}m_q \cos(l_p\alpha + m_q\beta) + \sum_{pq}\sum b_{pq}m_q \sin(l_p\alpha + m_q\beta) \qquad (13)$$

$$v/_{t=0} = y_t = -\sum_{pq}\sum a_{pq}l_p(\cos(l_p\alpha + m_q\beta) - \sum_{pq}\sum b_{pq}l_p \sin(l_p\alpha + m_q\beta)$$

and, since $x = \alpha$ and $y = \beta$, the initial conditions $u = u(x, y, 0)$, $v = v(x, y, 0)$ can be satisfied either in terms of a Fourier integral for an infinite domain or in terms of a Fourier series for a rectangular domain—or in terms of a two-variable stationary random process in an infinite domain. Whatever form of solution is chosen there will be a spectrum of wave numbers $k_{pq}^2 = l_p^2 + m_q^2$ such that the high-wave-number velocities will die out most rapidly.

At $t = 0$, suppose that the fluid particles bounded by the square: $x = \alpha = 0$, $a < y = \beta < 1$; $x = \alpha = 1$, $0 < y = \beta < 1$; $0 < x = \alpha < 1$, $y = \beta = 0$; and $0 < x = \alpha < 1$, $y = \beta = 1$ are dyed black. The motion of this dye patch can then be traced as time increases. For example, the edge of the square given by $\alpha = 0$, $0 < \beta < 1$ at $t = 0$ is given at time t_1 by

$$x = Vt_1 + \sum_{pq}\sum a_{pq}m_q \cdot \frac{(1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t_1])}{(\mu/\rho)(l_p^2 + m_q^2)} \cdot \cos(m_q\beta) + \sum_{pq}\sum b_{pq}m_q \cdot \frac{(1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t_1])}{(\mu/\rho)(l_p^2 + m_q^2)} \cdot \sin(m_q\beta) \qquad (14)$$

$$\begin{aligned} x = \alpha + Vt &+ \sum_{pq}\sum \frac{a_{pq}m_q(1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t])}{(\mu/\rho)(l_p^2 + m_q^2)} \cos(l_p\alpha + m_q\beta) \\ &+ \sum_{pq}\sum \frac{b_{pq}m_q(1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t])}{(\mu/\rho)(l_p^2 + m_q^2)} \sin(l_p\alpha + m_q\beta) \\ y = \beta &- \sum_{pq}\sum \frac{a_{pq}l_p(1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t])}{(\mu/\rho)(l_p^2 + m_q^2)} \cos(l_p\alpha + m_q\beta) \\ &- \sum_{pq}\sum \frac{b_{pq}l_p(1 - \exp[-(\mu/\rho)(l_p^2 + m_p^2)t])}{(\mu/\rho)(l_p^2 + m_q^2)} \sin(l_p\alpha + m_q\beta) \end{aligned} \qquad (12)$$

$$y = \beta - \sum_{pq} \sum a_{pq} l_p \cdot \frac{(1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t_1])}{(\mu/\rho)(l_p^2 + m_q^2)} \cdot \cos(m_q\beta) - \sum_{pq} \sum b_{pq} l_p \cdot \frac{(1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t_1])}{(\mu/\rho)(l_p^2 + m_q^2)} \cdot \sin(m_q\beta)$$

as β is varied from 0 to 1, since the two equations are the parametric representation of a curve $y = y(x)$ in the x,y plane.

Some trial solutions have been constructed for (14), and it appears possible to represent deformations almost as complex as those given by *Welander* [1955]. For small times the results show that the edges of the square are rippled by many high-wave-number irregularities. As time increases, the contributions from wave numbers corresponding to wavelengths the length of the sides of the square and longer appear, and the square can become highly deformed.

Although, by Stewart's definition, this motion is not turbulence, it does represent stirring. The perimeter of the dye patch, which is 4 units long at $t = 0$, increases manyfold as time increases. The area into which the dye could spread by molecular diffusion also increases manyfold as time increases.

However, this solution (and the solutions to follow) is not completely realistic, as the equation of continuity has been linearized. Two different fluid particles, α_1,β_1 and α_2,β_2, can be at the same point at the same time, so that at some time, t_1, later

$$x = x(\alpha_1, \beta_1, t_1) = x(\alpha_2, \beta_2, t_1)$$

and

$$y = y(\alpha_1, \beta_1, t_1) = y(\alpha_2, \beta_2, t_1)$$

This, of course, is physically impossible. Second-order corrections obtained by substituting (12) into (10) and solving (10) might provide even more realistic results.

The consequences of the failure to satisfy the equation of continuity exactly in such a system are difficult to comment upon since most work in hydrodynamics considers this equation one not to be trifled with. Nevertheless, such solutions may preserve certain other features of the fluid motion of greater importance.

It does not seem that any great physical significance should be placed upon this feature of the solution. *Miche* [1944], in studies of gravity waves by means of similar procedures, has evaluated the continuity equation to one order higher than the solution obtained. The result is some time-varying terms at the next higher order that serve as a way to judge how well the solution satisfies the equation of continuity. A variation of ± 0.10 seemed tolerable in the results of Miche.

Three-dimensional motions near a free surface. Consider solutions to equations 8 of the form given by (15), where the eigenvalues, λ, are to be found and A, B, C, and D can be complex.

$$\begin{aligned} z_1 &= Ae^{\lambda\delta}e^{i(l\alpha + m\beta - \omega t)} \\ x_1 &= Be^{\lambda\delta}e^{i(l\alpha + m\beta - \omega t)} \\ y_1 &= Ce^{\lambda\delta}e^{i(l\alpha + m\beta - \omega t)} \\ p_1/\rho &= De^{\lambda\delta}e^{i(l\alpha + m\beta - \omega t)} \end{aligned} \tag{15}$$

These equations yield

$$\begin{aligned} &(-\omega^2 + (i\omega\mu/\rho)(\lambda^2 - k^2))B + gilA + ilD = 0 \\ &(-\omega^2 + (i\omega\mu/\rho)(\lambda^2 - k^2))C + gimA + imD = 0 \\ &(-\omega^2 + g\lambda + (i\omega\mu/\rho)(\lambda^2 - k^2))A + \lambda D = 0 \\ &ilB + imC + \lambda A = 0 \end{aligned} \tag{16}$$

where $k^2 = l^2 + m^2$.

The determinant of the system of linear homogeneous equations must vanish for solutions to exist, and this determinant yields equation 17 for the eigenvalues.

$$\lambda^4 + \left(-2k^2 + \frac{i\rho\omega}{\mu}\right)\lambda^2 + \left(k^4 - \frac{i\rho\omega k^2}{\mu}\right) = 0 \tag{17}$$

The values of λ^2 become

$$\lambda^2 = k^2 - (i\rho\omega/2\mu) \pm \sqrt{-\rho^2\omega^2/\mu^2} \tag{18}$$

or

$$\lambda^2 = k^2 \tag{19}$$

$$\lambda^2 = k^2 - i\rho\omega/\mu$$

The four eigenvalues are therefore given by

$$\lambda = k \tag{20}$$

$$\lambda = -k$$

$$\lambda = k\left[\left(\frac{\theta+1}{2}\right)^{1/2} - i\left(\frac{\theta-1}{2}\right)^{1/2}\right]$$

$$\lambda = -k\left[\left(\frac{\theta+1}{2}\right)^{1/2} - i\left(\frac{\theta-1}{2}\right)^{1/2}\right]$$

where

$$\theta = [1 + (\rho^2\omega^2/\mu^2k^4)]^{1/2}$$

If a semi-infinite fluid with a free surface is now studied subject to the conditions that x, y, and z approach zero as δ approaches $-\infty$, the eigenvalue $\lambda = k$ yields

$$A = A_1 \tag{21}$$

$$B = ilA_1/k$$

$$C = imA_1/k$$

$$D = [(\omega^2/k) - g]A_1$$

For the eigenvalue $\lambda = k(r_1 - ir_2)$, where $r_1 = ((1+\theta)/2)^{1/2}$ and $r_2 = ((\theta-1)/2)^{1/2}$, the result is

$$A = A_2 \tag{22}$$

$$ilB + imC = -k(r_1 - ir_2)A_2$$

$$D = -gA_2$$

and, subject to the condition that the disturbance is oriented in the same direction of propagation as that of the previous solution, the additional condition that $B/l = C/m$ yields

$$A = A_2 \tag{23}$$

$$B = (il/k)(r_1 - ir_2)A_2$$

$$C = (im/k)(r_1 - ir_2)A_2$$

$$D = -gA_2$$

The pressure is given by

$$\frac{p}{\rho} = \left(\frac{\omega^2}{k} - g\right)A_1e^{k\delta}e^{i(l\alpha+m\beta-\omega t)} - gA_2e^{k(r_1-ir_2)\delta}e^{i(l\alpha+m\beta-\omega t)} \tag{24}$$

The pressure at the free surface, $\delta = 0$, can be zero if $A_2 = 0$ and $\omega^2 = gk$, which is the condition for gravity waves, or it can be zero if

$$A_2 = [(\omega^2/gk) - 1]A_1 \tag{25}$$

If this value of A_2 is taken, if A_1 is chosen real, and if m is set equal to zero so that $l = k$, the real form of the solution is given by

$$x_1 = A_1\Big[-e^{k\delta}\sin(k\alpha - \omega t) - r_1\left(\frac{\omega^2}{gk} - 1\right)e^{kr_1\delta}\sin(-kr_2\,\delta + k\alpha - \omega t) + r_2\left(\frac{\omega^2}{gk} - 1\right)e^{kr_1\delta}\cos(-kr_2\,\delta + k\alpha - \omega t)\Big] \tag{26}$$

$$y_1 = 0$$

$$z_1 = A_1\Big[e^{k\delta}\cos(k\alpha - \omega t) + \left(\frac{\omega^2}{kg} - 1\right)e^{kr_1\delta}\cos(-kr_2\,\delta + k\alpha - \omega t)\Big]$$

$$\frac{p_1}{\rho} = A_1\Big[\left(\frac{\omega^2}{k} - g\right)e^{k\delta}\cos(k\alpha - \omega t) - \left(\frac{\omega^2}{k} - g\right)e^{kr_1\delta}\cos(-kr_2\,\delta + k\alpha - \omega t)\Big]$$

A second interesting solution is found by finding the solution similar to (26) except that $l = -k$ and adding it to (26). The result is a cellular pattern given by equation 27 along with the appropriate pressure equation.

$$x_1 = 2A_1\Big[-e^{k\delta}\cos\omega t - r_1\left(\frac{\omega^2}{gk} - 1\right)e^{kr_1\delta}\cos(kr_2\,\delta + \omega t) + r_2\left(\frac{\omega^2}{gk} - 1\right)e^{kr_1\delta}\sin(kr_2\,\delta + \omega t)\Big]\sin k\alpha \tag{27}$$

$$z_1 = 2A_1\Big[e^{k\delta}\cos\omega t + \left(\frac{\omega^2}{gk} - 1\right)e^{kr_1\delta} \cdot \cos(kr_2\,\delta + \omega t)\Big]\cos k\alpha$$

The solutions given by (26) and (27) are rather difficult to understand in their Lagrangian form. Three types of motion seem to be of interest. The first type occurs when $\omega^2 = gk$.

Equation 26 then reduces to the form for gravity wave motion in Lagrangian form, and, in particular, the Gerstner wave is a result. The terms involving $(\mu/\rho)\nabla_L^2$ vanish identically. Equation 27 is the corresponding standing-wave solution.

If, for the second type of solution, ω^2 is approximately equal to gk, we can write

$$\theta \cong \frac{\rho^2\omega^2}{\mu^2k^4} \cong \frac{\rho^2 g}{\mu^2k^3} \cong \frac{6 \times 10^6}{k^3} \tag{28}$$

so that

$$r_1k \cong r_2k \cong 35k^{1/4} \tag{29}$$

As k varies from 1 to 0.0001, and the wavelength correspondingly varies from 6.28 cm to 628 meters, r_1k varies from 35 to 3.5.

If $\omega^2 = 1.1gk$, so that the frequency is not quite appropriate to a gravity wave motion, $r_1[(\omega^2/gk)-1]$ is approximately equal to $3.5/k^{3/4}$ and ranges from 3.5 to 3500 as k varies from 1 to 0.0001.

The vertical motion in (26) for $\omega^2 = 1.1gk$ is only slightly increased. The increase dies out rapidly with depth within a fraction of a centimeter although it oscillates in direction rapidly with depth. The horizontal motion at the surface is greatly increased. It also dies out rapidly with depth and oscillates in direction with depth. The very strong shear with depth and the strong velocities that result would probably make the resulting motions unstable. Thus what might be called decoupled gravity waves are probably not frequent in nature.

The third type of solution is probably the most interesting. There is no reason in (26) and (27) why ω^2 has to be anywhere near the value gk. If, for example, ω^2 is set equal to $gk\ 10^{-4}$, so that the periods of the resulting motion will be 100 times longer than the corresponding gravity wave periods for corresponding wavelengths, we can write

$$\theta \cong \frac{\rho^2 gk}{\mu^2k^4}\,10^{-4} \cong \frac{600}{k^3} \tag{30}$$

and

$$r_1k \cong r_2k \cong 3.5k^{1/4} \tag{31}$$

As k varies from 1 to 0.001, r_1k varies from 3.5 to 0.35 and r_1 varies from 3.5 to 3500. The term ω^2/gk is negligible compared with 1.

Thus for k equal to 1, for example, the new terms contribute motions about 5 times that of the wavelike term in the horizontal and comparable to the wavelike term in the vertical. The vertical motion at the surface is very small, essentially zero. The horizontal scale of the motion is about 6.28 cm. The vertical scale is a combination of scales of 6.28 cm and 1.80 cm.

The properties of equations 26 and 27 have not yet been analyzed fully, but the author is of the opinion that equation 27 in particular looks very much like what an 'eddy' ought to look like under these conditions and that a proper superposition of solutions in the form of (26) carried out to second order might give some understanding of how heat and dyes can be spread so rapidly in the upper layers of the ocean compared with what would be computed from laminar solutions of the Eulerian form of the equations.

Decaying random motions. Another solution to the linear equations is given by equations 32.

$$\begin{aligned}
x &= \alpha \\
&\quad + \sum_q \sum_r \frac{[1 - \exp[-(\mu/\rho)(m_q^2 + n_r^2)t]]}{(\mu/\rho)(m_q^2 + n_r^2)} \\
&\quad \cdot [A_{qr} \cos(m_q\beta + n_r\,\delta) \\
&\quad + B_{qr} \sin(m_q\beta + n_r\,\delta)] \\
y &= \beta \\
&\quad + \sum_p \sum_r \frac{[1 - \exp[-(\mu/\rho)(l_p^2 + n_r^2)t]]}{(\mu/\rho)(l_p^2 + n_r^2)} \\
&\quad \cdot [C_{pr} \cos(l_p\alpha + n_r\,\delta) \\
&\quad + D_{pr} \sin(l_p\alpha + n_r\,\delta)] \\
z &= \delta \\
&\quad + \sum_q \sum_p \frac{[1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t]]}{(\mu/\rho)(l_p^2 + m_q^2)} \\
&\quad \cdot [E_{pq} \cos(l_p\alpha + m_q\beta) \\
&\quad + F_{pq} \sin(l_p\alpha + m_q\beta)] \\
p &= p_0 - g\rho\,\delta - g\rho \\
&\quad \cdot \Bigg[\sum_q \sum_r \frac{[1 - \exp[-(\mu/\rho)(l_p^2 + m_q^2)t]]}{(\mu/\rho)(l_p^2 + m_q^2)} \\
&\quad \cdot [E_{pq} \cos(l_p\alpha + m_q\beta) \\
&\quad + F_{pq} \sin(l_p\alpha + m_q\beta)] \Bigg]
\end{aligned} \tag{32}$$

The constants A_{qr}, B_{qr}, C_{pr}, D_{pr}, E_{pq}, and F_{pq} can be chosen as independent random variables from normal distributions with variances assigned in an appropriate limiting process by power spectra in such a way that $x - \alpha$, $y - \beta$, and $z - \delta$ when paired as functions of either α, β, or δ (as appropriate) would be incoherent.

At $t = 0$, $x = \alpha$, $y = \beta$, and $z = \delta$, and u, v, and w are obtained by computing x_t, y_t, and z_t and substituting x, y, and z for α, β, and δ. Thus

$$u = \sum_q \sum_r A_{qr} \cos (m_q y + n_r z) + B_{qr} \sin (m_q y + n_r z) \tag{33}$$

$$v = \sum_p \sum_r C_{pr} \cos (l_p x + n_r z) + D_{pr} \sin (l_p x + n_r z)$$

$$w = \sum_p \sum_q E_{pq} \cos (l_p x + m_q y) + F_{pq} \sin (l_p x + m_p y)$$

and so each component of the vorticity is irregular, aperiodic, and incoherent, but stationary, in space at $t = 0$.

The fluid particle at $\alpha = \beta = \delta = 0$ at $t = 0$ is at

$$x_{\mathrm{I}} = \sum_q \sum_r A_{qr}^* \tag{34}$$

$$y_{\mathrm{I}} = \sum_p \sum_r C_{pr}^*$$

$$z_{\mathrm{I}} = \sum_p \sum_q E_{pq}^*$$

after a long time has passed. The particle at $\alpha = 1$, $\beta = \delta = 0$, is at

$$x_{\mathrm{II}} = 1 + \sum_q \sum_r A_{qr}^* \tag{35}$$

$$y_{\mathrm{II}} = \sum_p \sum_r C_{pr}^* \cos l_p + D_{pr}^* \sin l_p$$

$$z_{\mathrm{II}} = \sum_p \sum_q E_{pq}^* \cos l_p + D_{pq}^* \sin l_p$$

after a long time has passed. Two fluid particles originally 1 unit apart can end up quite far apart. (In equations 34 and 35, $A_{qr}{}^*$ equals A_{qr} $((\mu/\rho)\ (m_q{}^2 + n_r{}^2))^{-1}$ and so on.)

This solution illustrates two of the three requirements set forth by Stewart for turbulence. No transfer of energy from larger to smaller scales appears to be exhibited in this solution, because it is linear. This has to be checked in an Eulerian reference system, as strange things can happen. For example, if we were to try to recover $u = u(x, y, z, t)$, $v = v(x, y, z, t)$, and $w = w(x, y, z, t)$ from (32) and its derivatives with respect to time, higher-frequency interactions would appear that might look like nonlinear interactions. However, with two requirements already fulfilled, there is a possibility that either further analysis or finding the second- (or third-) order solution will exhibit this third desired feature.

Other possible linear solutions. Other possible linear solutions might be found that would represent sustained irregular flow, given a randomized source of energy for the system. Extension to compressible flows would not be difficult, and the eddy conduction of heat might be a problem that could be treated. The remarks of *Kampé de Fériet* [1961, p. 111] are pertinent here. Upon consideration of the amount of effort that has gone into the study of the Lagrangian equations compared with the study of the Eulerian equations, it would appear that much could be learned from pursuing the path described above.

Higher-order solutions. Once the linear problem has been formulated, substitution into the higher-order equations produces higher-order modifications to the original linear problem. Sum and difference frequencies appear in the second-order solution, and perhaps certain restrictions on possible wave numbers and admissible frequencies will result. In a randomized gravity wave theory developed along these lines, the second-order solution yielded the well-known change in phase speed that arises when the Eulerian equations are studied to third order [*Pierson*, 1961]. Thus certain nonlinear features of higher order in the Eulerian system appear at lower order in the Lagrangian system.

To carry out any of the results given above to second or third order requires a great deal of algebraic manipulation. In time, perhaps this will be done and new insight into fluid motions will be obtained.

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