JOURNAL OF GEOPHYSICAL RESEARCH VOLUME 67, No. 8 JULY 1962

Integral Diffusivity[1]

J. C. SCHÖNFELD

Physical Division, Rijkswaterstaat, The Hague, Netherlands

Abstract. In oceanography much attention has been paid to the influence of the scale factor in diffusion processes. Recently the problem has been approached with the aid of a Fourier spectrum representation of diffusible property, and by relating the diffusivity to the wavelength parameter. This so-called 'integral diffusion concept' is now being improved by making allowance for the persistence in time of the turbulent eddies active in the diffusion. The resulting mathematical discipline is also developed from a more axiomatic point of view.

1. *Introduction.* Diffusion in a medium like the ocean is caused predominantly by turbulent motion. The older theories of turbulent fluid motion were generally modeled after the analytical description of molecular viscosity. The problem of turbulent shear flow, for instance, has been treated by means of a turbulent viscosity. This viscosity does not appear to be constant. The introduction of the mixing length concept has improved this mathematical description. The analytical approach to turbulent fluid motion problems, however, has never reached the stage of a general theory.

In more recent investigations the application of statistical methods has gradually come to the fore. Increasing empirical knowledge by ever more refined measuring techniques, and the demand for statistical evaluation of the measurements, have stimulated the growing emphasis on statistics.

The incomplete success of analytical methods, on the other hand, becomes more understandable when we consider that the mathematical treatment of the basic equations of fluid motion (the Navier-Stokes equation, for example) is complicated by their nonlinear character. Yet these nonlinearities seem to be an essential agency in the generation of turbulance.

The situation is different in problems of turbulent diffusion, provided that we confine ourselves to diffusion in fields of autonomous turbulence. By this we understand that the turbulent motions follow their own laws, not being influenced by the variable concentration of the diffusate. Then the diffusion forms an essentially linear problem, permitting us to proceed by analytical methods, such as those inaugurated by *Kolmogoroff* [1931].

Initially, turbulent diffusion was treated by the classical Fickian equation. The diffusion, however, depends on the scale of the predominant eddies. Since in different situations different eddies may be dominating, the diffusivity cannot be a constant as in Fick's conception. Hence it has been attempted to make the turbulent diffusivity variable, depending, for instance, on the place, as has been done for turbulent viscosity in shear flow.

The example of stationary, homogeneous turbulence, however, demonstrates the inadequacy of such an approach. Any characteristic of stationary, homogeneous turbulence must be independent of place and time. The same must also be true for the diffusivity, whereas, on the other hand, a constant value for the diffusivity ignores the scale effect.

An ingenious effort to solve the question was made by *Richardson* [1926], who introduced the idea of neighbor diffusivity and postulated his neighbor diffusion equation. This concept fulfils the requirement that the diffusivity depends on a scale measure, the neighbor separation parameter, and yet not on place and time. Therefore the neighbor diffusion hypothesis has been a very useful instrument not only in meteorology but also in oceanography (cf. *Stommel* [1949]).

Richardson's hypothesis nevertheless has some important limitations, as we shall discuss more

[1] Based on a paper presented at the International Symposium on Fundamental Problems in Turbulence and Their Relation to Geophysics sponsored by the International Union of Geodesy and Geophysics and the International Union of Theoretical and Applied Mechanics, held September 4–9, 1961, in Marseilles, France.

fully in section 6. This may explain why in several more recent publications space- and time-dependent diffusivities are still being used [*Joseph and Sendner*, 1958; *Ozmidow*, 1958].

Together with these more analytical treatments of turbulent diffusion, the statistical investigation of diffusion processes is advancing. From the statistical point of view, diffusivity can be defined by the time variation of statistical parameters describing a cloud of diffusate. An apparent diffusivity may, for instance, be derived from the time derivative of the variance of the radius vector in the cloud [*Frenkiel*, 1953].

In the present investigation we have attempted a more analytical approach, using Fourier transformations and exploiting the linear character of the diffusion problem. Diffusivity is defined by a spectrum function in such a way that classical diffusivity is included as a special case.

2. *The integral diffusion concept.* Let us consider the diffusion of some physical property by a two-dimensional, stationary, homogeneous, and isotropic turbulent field. Let the local concentration of the diffusate be $s(t, x, y)$. This should be interpreted as a statistical average (see section 4). Let (u_x, u_y) denote the vector of transport by turbulent diffusion.

In the classical, Fickian theory of diffusion transport is defined by the local gradient of the concentration:

$$u_x = K \frac{\partial s}{\partial x} \qquad u_y = K \frac{\partial s}{\partial y} \tag{1}$$

The coefficient K is called the diffusivity.

This approach seems justified when the paths of the irregular diffusion movements are small compared with the dimensions of the cloud of diffusate. The paths of molecular movement usually satisfy this condition. In turbulent movement, however, the paths may be on such a large scale that the transport should be supposed to depend not so much on the local gradient of the concentration as on the distribution of the concentration in a larger area. We can then argue as follows:

Consider an 'eddy' characterized by a length measure ρ, engaged at the point (x, y). The eddy is assumed to produce a velocity w in a direction at an angle χ with the x axis. Then the concentration at (x, y) will incidentally deviate from the statistic mean value $s(x, y)$. We estimate this deviating value by

$$s(x - \rho \cos \chi, y - \rho \sin \chi)$$

Then the transport will be

$$s(x - \rho \cos \chi, y - \rho \sin \chi)w$$

in the direction χ, with a component

$$s(x - \rho \cos \chi, y - \rho \sin \chi)w \cos \chi$$

in the x direction.

Here it is assumed that the diffusive transport depends on the instantaneous distribution. Although this is true in classical diffusion, it seems more questionable in the case here discussed. For the time being, however, we ignore the objection.

To find the resulting transport u_x, the above expression has to be averaged statistically for all possible eddies. Let $w^* \, d\rho/\rho$ be the statistic weight of the eddy with length ρ. We assume, moreover, that all directions χ are equally probable. Then

$$u_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi \int_0^\infty \frac{d\rho}{\rho} \omega(\rho) \cdot s(x - \rho \cos \chi, y - \rho \sin \chi) \cos \chi \tag{2}$$

where the weighted turbulent velocity

$$\omega = ww^*$$

is assumed to be a function of ρ only.

Next we apply a Fourier transform to s:

$$S(t, \lambda, \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \, s(t, x, y) e^{-2\pi i (\lambda x + \mu y)} \tag{3}$$

In a similar way transforms U_x and U_y can be associated with u_x and u_y.

Then by putting

$$K(\sigma) = \frac{1}{2\pi\sigma} \int_0^\infty \frac{d\rho}{\rho} \omega(\rho) J_1(2\pi\rho\sigma) \tag{4}$$

where J_1 denotes the Bessel function of the first order, we can reduce (2) to an involution integral from which follows an associated equation for U_x:

$$U_x = -2\pi i\lambda K(\sigma) S \qquad U_y = -2\pi i\mu K(\sigma) S \tag{5}$$

The equation for U_y has been supplied by analogy.

We arrive at similar equations by applying the Fourier transformation to (1). Only K is then a constant.

In molecular diffusion we may suppose that ω is large for very small values of ρ and negligibly small for values of ρ greater than a few times the molecular free path, say greater than ϵ. Then for $\sigma < 1/\epsilon$ we have approximately

$$K(\sigma) \approx \int_0^\infty d\rho\, \omega(\rho) = \text{constant}$$

so that we return to the classical diffusion concept.

This will also hold good for turbulent diffusion when the prevailing eddies are small compared with the dimensions of the cloud of diffusate.

The definition (2) which replaces (1) suggests that $K(\sigma)$ be called the *integral diffusivity.*

To provide a further interpretation of (5), we consider the inverse transformation

$$s(t, x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda\, d\mu\, S(t, \lambda, \mu) e^{2\pi i(\lambda x + \mu y)} \qquad (6)$$

which, by introducing polar coordinates r, φ in the xy plane and σ, ψ in the $\lambda\mu$ plane, takes the form

$$s(t, r, \varphi) = \int_0^\infty \sigma\, d\sigma \cdot \int_{-\pi}^{\pi} d\psi\, S(t, \sigma, \psi) e^{2\pi i\sigma r \cos(\psi - \varphi)} \qquad (7)$$

When s is real, the complex function $S(t, \sigma, \psi+\pi)$ is conjugate to $S(t, \sigma, \psi)$, so that (7) can be further reduced to

$$s(t, r, \varphi) = \int_0^\infty \sigma\, d\sigma \int_0^\pi d\psi\, 2\,|S| \cdot \cos\{2\pi\sigma r \cos(\psi - \varphi) + \arg S\} \qquad (8)$$

Now $\cos\{2\pi\sigma r \cos(\psi - \varphi) + \arg S\}$ represents a periodic function in the xy plane in the form of a 'wave,' with the crest lines normal to the direction $\varphi = \psi$, and the wavelength $L = 1/\sigma$. Hence (8) expresses the distribution s by superposition of sine waves with variable wave number σ and variable direction ψ.

If we consider a wave with wave number σ, it is plausible that this wave is submitted predominantly to the diffusive action of eddies with dimensions of the same order of magnitude as the wavelength $L = 1/\sigma$ [*Groen,* 1954], so that the diffusivity becomes a function of σ, as we see by (5).

In the absence of an advective mean flow, the transport equation in two dimensions takes the form

$$\frac{\partial s}{\partial t} + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = q \qquad (9)$$

where $q(t, x, y)$ is the rate at which diffusate is released into the field per unit time and per unit area. Substitution from (5) yields

$$\frac{\partial S}{\partial t} + 4\pi^2\sigma^2 K(\sigma) S = Q(t, \lambda, \mu) \qquad (10)$$

where Q is associated with q by Fourier transformation.

The definition of diffusivity here arrived at observes on the one hand the dependence on the scale of the turbulent eddies, whereas on the other hand the diffusivity is independent of time and place, just as it should be in a stationary, homogeneous field.

When the turbulent field is no longer supposed to be isotropic, we have to assume that ω also depends on the angle χ: $\omega = \omega(\rho, \chi) = \omega(\xi, \eta)$. Here $\xi = \rho \cos \chi$ and $\eta = \rho \sin \chi$.

The contents of this section summarize the results of a previous report [*Schönfeld,* 1959], which was communicated to the IUGG congress at Helsinki in 1960.

3. *The time scale of integral diffusion.* Now we return to the objection raised during the derivation of (2). In this derivation we accounted for the finite length dimensions of the eddies. It is reasonable, however, to account likewise for a finite delay in their diffusive action. Hence we amend (2) as follows:

$$u_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\xi\, d\eta}{\rho^2} \frac{\xi}{\rho} \int_0^\infty \frac{d\tau}{\tau} \omega(\tau, \xi, \eta) \cdot s(t - \tau, x - \xi, y - \eta) \qquad (11)$$

It is to be expected that the time of delay τ is correlated to the space shift ρ, but discussion of this point of view will be postponed until section 5.

We introduce a further Fourier transform with respect to time, so that we obtain

$$\text{S}(f, \lambda, \mu) = \int_{-\infty}^{\infty} dt S(t, \lambda, \mu) e^{-2\pi i f t} \qquad (12)$$

and similarly U_x, U_y, and Q. Then, by putting

$$K(f, \lambda, \mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi\, d\eta\, i\,\frac{\lambda\xi + \mu y}{4\pi^2\sigma^2\rho^3}\, e^{-2\pi i(\lambda x+\mu y)} \cdot \int_{-\infty}^{\infty} \frac{d\tau}{\tau}\, e^{-2\pi i f\tau}\omega(\tau, \xi, \eta) \qquad (13)$$

we arrive at

$$2\pi i f\ S + 4\pi^2\sigma^2 K(f, \lambda, \mu)\ S = Q \qquad (14)$$

instead of (10).

When there is an advective mean flow defined by the vector (v_x, v_y), the transport equation becomes

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}(u_x + sv_x) + \frac{\partial}{\partial y}(u_y + sv_y) = q \qquad (15)$$

Then the Fourier transform (14) is amended as follows:

$$2\pi i f\ S(f, \lambda, \mu) + 4\pi^2\sigma^2\ K(f, \lambda, \mu)S(f, \lambda, \mu) + 2\pi i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda_1\, d\mu_1 \cdot \int_{-\infty}^{\infty} df_1[\lambda\ V_x(f_1, \lambda_1, \mu_1) + \mu\ V_y(f_1, \lambda_1, \mu_1)] \cdot S(f - f_1, \lambda - \lambda_1, \mu - \mu_1) = Q(f, \lambda, \mu) \qquad (16)$$

where V_x and V_y are associated with v_x and v_y by Fourier transformation.

When the supposition of stationary, homogeneous turbulence is dropped, we may assume that ω also depends on time and place: $\omega = \omega(t, x, y, \tau, \xi, \eta)$. Let $\Omega(f, \lambda, \mu, \tau, \xi, \eta)$ be the Fourier transform of ω with respect to t, x, and y, and let

$$K(f, \lambda, \mu, f_1, \lambda_1, \mu_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi\, d\eta\, i\,\frac{\lambda\xi + \mu\eta}{4\pi^2\sigma^2\rho^3}\, e^{-2\pi i(\lambda_1\xi+\mu_1\eta)} \cdot \int_{-\infty}^{\infty} \frac{d\tau}{\tau}\, e^{-2\pi i f_1\tau} \cdot \Omega(f - f_1, \lambda - \lambda_1, \mu - \mu_1, \tau, \xi, \eta)$$

Then the product KS in (16) has to be replaced by the involution integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda_1\, d\mu_1 \cdot \int_{-\infty}^{\infty} df_1\, K(f, \lambda, \mu, f_1, \lambda_1, \mu_1)\ S(f_1, \lambda_1, \mu_1) \qquad (17)$$

This means that in the diffusive transport in waves with a definite wavelength, direction, and period there may be interference by waves with other wavelengths, etc.

We have here arrived by induction at what we believe to be the most general Eulerian description of the mean concentration of diffusate in an infinite two-dimensional turbulent field. Reduction to one dimension and extension to three dimensions are obvious.

4. *Deductive arguments.* In the preceding sections we have developed the integral diffusion concept by inductive arguments. A more deductive approach is followed in this section.

The transport equation in a two-dimensional continuous medium is

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}(v_x s) + \frac{\partial}{\partial y}(v_y s) - K_m\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}\right) = q \qquad (18)$$

Here K_m denotes the molecular diffusivity. The flow vector (v_x, v_y) is assumed to be a function of t, x, and y, and moreover of one or more statistic parameters, condensed by the symbol p: $v_x = v_x(t, x, y, p)$.

We suppose that the concentration of diffusate does not affect the velocity field.

Even the smallest eddies in the turbulent field, controlled by viscosity, are supposed to have such dimensions that a great number of molecules is involved in a single eddy. The assumption that the medium is continuous is then justified.

The mean flow can be defined by

$$\bar{v}_x(t, x, y) \cdot \int dp = \int dp\, v_x(t, x, y, p) \qquad (19)$$

and a similar expression for $\bar{v}_y$. Here the integrations are extended to the whole region of possible values of the set of parameters p.

The distribution must also depend on the statistic parameters: $s = s(t, x, y, p)$. Averaging with respect to p yields the mean distribution $\bar{s}$, which was denoted by s in the preceding section.

We assume that all the diffusate present in the field has been introduced by the releasing function q.

We apply Fourier transformations with respect to t, x, and y, so that functions S, V_x, V_y, and Q are associated with s, v_x, v_y, and q. We suppose that the behavior of s, etc., for $t = \pm\infty$ and $r = \infty$ is such that the Fourier integrals are convergent. Then (18) reduces to

$$2\pi i f \ S(f, \lambda, \mu, p) + 4\pi^2 K_m \sigma^2 \ S(f, \lambda, \mu, p) + 2\pi i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda_1 \, d\mu_1 \int_{-\infty}^{\infty} df_1 \cdot [\lambda V_x(f_1, \lambda_1, \mu_1) + \mu V_y(f_1, \lambda_1, \mu_1)] \cdot S(f - f_1, \lambda - \lambda_1, \mu - \mu_1, p) = Q(f, \lambda, \mu) \tag{20}$$

When Q, V_x, and V_y are supposed to be known, (20) forms an integral equation for S.

When we introduce

$$A(f, \lambda, \mu, f_2, \lambda_2, \mu_2, p) = (2\pi i f + 4\pi^2 K_m \sigma^2) \cdot \delta(f - f_2) \, \delta(\lambda - \lambda_2) \, \delta(\mu - \mu_2) + 2\pi i \{\lambda V_x(f - f_2, \lambda - \lambda_2, \mu - \mu_2, p) + \mu V_y(f - f_2, \lambda - \lambda_2, \mu - \mu_2, p)\}$$

we can rewrite (20) as follows:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda_2 \, d\mu_2 \int_{-\infty}^{\infty} df_2 \cdot A(f, \lambda, \mu, f_2, \lambda_2, \mu_2, p) \cdot S(f_2, \lambda_2, \mu_2, p) = Q(f, \lambda, \mu)$$

When S can be solved from this equation, the solution must be of the form

$$S(f, \lambda, \mu, p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda_3 \, d\mu_3 \int_{-\infty}^{\infty} df_3 \cdot B(f, \lambda, \mu, f_3, \lambda_3, \mu_3, p) \, Q(f_3, \lambda_3, \mu_3) \tag{21}$$

Here B must satisfy the equation

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda_2 \, d\mu_2 \int_{-\infty}^{\infty} df_2 \ A(f, \lambda, \mu, f_2, \lambda_2, \mu_2, p) \cdot B(f_2, \lambda_2, \mu_2, f_3, \lambda_3, \mu_3, p) = \delta(f - f_3) \, \delta(\lambda - \lambda_3) \, \delta(\mu - \mu_3) \tag{22}$$

where δ denotes the Dirac impulse function. As A is defined by the spectra V_x and V_y of the turbulent field, B is likewise defined by this field.

Averaging (21) statistically yields

$$\bar{S}(f, \lambda, \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda_3 \, d\mu_3 \int_{-\infty}^{\infty} df_3 \cdot \bar{B}(f, \lambda, \mu, f_3, \lambda_3, \mu_3) \, Q(f_3, \lambda_3, \mu_3)$$

When this is inverted we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda_2 \, d\mu_2 \int_{-\infty}^{\infty} df_2 \ C(f, \lambda, \mu, f_2, \lambda_2, \mu_2) \cdot \bar{S}(f_2, \lambda_2, \mu_2) = Q(f, \lambda, \mu) \tag{23}$$

where C must satisfy the equation

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda_2 \, d\mu_2 \int_{-\infty}^{\infty} df_2 \ C(f, \lambda, \mu, f_2, \lambda_2, \mu_2) \cdot \bar{B}(f_2, \lambda_2, \mu_2, f_3, \lambda_3, \mu_3) = \delta(f - f_3) \, \delta(\lambda - \lambda_3) \, \delta(\mu - \mu_3) \tag{24}$$

We can reduce (23) to (16) as amended by (17) if we put

$$2\pi i f \ \delta(f - f_2) \, \delta(\lambda - \lambda_2) \, \delta(\mu - \mu_2) + 4\pi^2 \sigma^2 K(f, \lambda, \mu, f_2, \lambda_2, \mu_2) + 2\pi i \{\lambda \bar{V}_x(f - f_2, \lambda - \lambda_2, \mu - \mu_2) + \mu \bar{V}_y(f - f_2, \lambda - \lambda_2, \mu - \mu_2)\} = C(f, \lambda, \mu, f_2, \lambda_2, \mu_2)$$

This may be interpreted as a definition of the function K from the given turbulent field.

When there is no mean flow and the turbulent field is stationary and homogeneous, the following reduction is possible:

In a stationary, homogeneous field the mean distribution $\bar{s}(t, x, y)$ and its generating function $q(t, x, y)$ are invariant to any translation X, Y in space, or to a shift T in the time.

A translation in space and a shift in time are accounted for by multiplication of $\bar{S}$ and Q by the factor

$$e^{-2\pi i (fT + \lambda X + \mu Y)}$$

Hence (23) yields

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} df_2 \, d\lambda_2 \, d\mu_2 C(f, \lambda, \mu, f_2, \lambda_2, \mu_2) \cdot e^{-2\pi i (f_2 T + \lambda_2 X + \mu_2 Y)} \, \bar{S}(f_2, \lambda_2, \mu_2) = e^{-2\pi i (fT + \lambda X + \mu Y)} \, Q(f, \lambda, \mu) \tag{25}$$

Now the left-hand member has the form of a Fourier transformation from f_2, λ_2, μ_2 into T, X, Y. Inversion of this transformation yields

$$C(f, \lambda, \mu, f_2, \lambda_2, \mu_2)\bar{S}(f_2, \lambda_2, \mu_2)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dT\, dX\, dY\, e^{2\pi i(f_2 T+\lambda_2 X+\mu_2 Y)}$$
$$\cdot e^{-2\pi i(fT+\lambda X+\mu Y)}\, Q(f, \lambda, \mu)$$
$$= Q(f_2, \lambda_2, \mu_2)\, \delta(f - f_2)\, \delta(\lambda - \lambda_2)\, \delta(\mu - \mu_2)$$

Hence C must here be of the form

$$C = R(f_2, \lambda_2, \mu_2) \cdot \delta(f - f_2)\, \delta(\lambda - \lambda_2)\, \delta(\mu - \mu_2)$$

whereas (25) reduces to

$$R(f, \lambda, \mu)\, \bar{S}(f, \lambda, \mu) = Q(f, \lambda, \mu)$$

This is identical with (14) if we put

$$R(f, \lambda, \mu) = 2\pi i f + 4\pi^2\sigma^2\, K(f, \lambda, \mu)$$

In this way we see that (14) can also be inferred statistically from the basic equation 18.

5. *Similarity.* We shall now consider in some more detail the structure of the diffusivity spectrum in stationary, homogeneous, and isotropic turbulence. We assume that there is no mean flow. Then (14) reduces to

$$2\pi i f\, S + 4\pi^2\sigma^2 K\, S = Q \tag{26}$$

where $K = K(f, \sigma)$.

So far no special assumption on the nature of the turbulent field has been admitted. It is well known, however, that turbulent fluid motion is generally dominated by patterns of motion in which there is a correlation between length and time scales. This involves a correlation between f and σ in the spectrum of the turbulent velocities.

It must be expected that a similar correlation will be found in the diffusivity spectrum $K(f, \sigma)$.

Hence, for a definite value of σ, f will be greatest around a value f_0 of f, which is a function of σ: $f_0 = f_0(\sigma)$.

The energy transfer from larger to smaller eddies per unit mass of medium will be of the order of magnitude

$$E = \sigma^{-2} f_0^3(\sigma)$$

In the Kolmogoroff range this energy transfer is constant. In a medium like the ocean, however, diffusion may also take place in a range of the spectrum where the supply of energy from external sources is significant. In that case E increases with σ, say proportional to σ^β approximately:

$$E = c^3\sigma^\beta$$

Hence

$$f_0 = c\sigma^{(2+\beta)/3} \tag{27}$$

If the supply of energy from external sources follows similar laws in different parts of the spectrum, the structure of the turbulence will also be similar. Then the diffusivity function K may be supposed to be defined by a relation of the form

$$\Phi[K/(\sigma^{-2}f_0),\, f/f_0] = 0 \tag{28}$$

By introducing the diffusion velocity

$$W = \sigma^{-1} f_0 = c\sigma^{-\alpha} \tag{29}$$

where $\alpha = (\beta - 1)/3$, (28) can also be formulated as

$$K = \frac{W}{2\pi\sigma}\, \mathrm{F}\left(\frac{f}{2\pi W\sigma}\right) \tag{30}$$

Let $F(t')$ represent the inverse Fourier transform of $\mathrm{F}(f')$. Then the inverse transformation of (26) with respect to f yields

$$\frac{\partial S}{\partial t} + (2\pi W\sigma)^2 \int_{-\infty}^{\infty} d\theta\, F(2\pi W\sigma\theta) \cdot S(t - \theta, \lambda, \mu) = Q(t, \lambda, \mu) \tag{31}$$

This equation gives rise to the following remark:

When no diffusate is introduced before the instant t_0 ($Q = 0$ for $t < t_0$), S must likewise be zero for $t < t_0$. This is not possible according to (31), unless

$$F(t') = 0 \quad \text{for} \quad t' < 0$$

In that case F can be interpreted as a retardation function of the diffusion, and the lower limit of the integral in (31) may be replaced by zero.

The definition of E and c may be so chosen that $F(0)$ becomes unity. Then

$$\int F(t')\, dt' = 1$$

likewise.

We now consider more in particular the distribution of $s_I(t, x, y)$ caused by releasing a unit quantity of diffusate at the origin at time zero. Then

$$q = \delta(t)\ \delta(x)\ \delta(y)$$

from which we deduce

$$Q = 1$$

Accordingly we deduce from (26), when substituting from (30),

$$S_I = \frac{1}{2\pi W\sigma} \cdot \frac{1}{2\pi i\ f/(2\pi W\sigma) + F(f/[2\pi W\sigma])} \quad (32)$$

Let

$$G(t') = \int_{-\infty}^{\infty} df' \frac{1}{2\pi i f' + F(f')} e^{2\pi i f' t'} \quad (33)$$

Then

$$S_I = G(2\pi W\sigma t) \quad (34)$$

Substituting from (29), the inverse transformation with respect to σ yields

$$s_I = (2\pi ct)^{-2/(1-\alpha)} H(r(2\pi ct)^{-1/(1-\alpha)}) \quad (35)$$

where

$$H(r') = \int_0^{\infty} 2\pi\sigma'\ d\sigma'\, J_0(2\pi\sigma' r') G((\sigma')^{1-\alpha}) \quad (36)$$

Hence the form of the unit release distribution s_I appears to be interrelated with the retardation function F.

When the unit release distribution s_I is known, the solution for an arbitrary release function $q(t, x, y)$ is found as follows:

From (32) and (26) we deduce that

$$S = Q\, S_I$$

This corresponds to the involution integral

$$s(t, x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1\ dx_1\ dy_1 \cdot q(t_1, x_1, y_1)\ s_I(t - t_1, x - x_1, y - y_1) \quad (37)$$

We may remark that, if we postulate (37), it can be shown that (14) must be valid. This is another way to establish this fundamental formula.

Until now we have confined ourselves to the two-dimensional case. In one- and three-dimensional isotropic media equation 14 remains fundamentally the same.

Let $s_I^{(m)}$ denote the m-dimensional unit release distribution. Then we require that the following reduction theorem be valid:

$$s_I^{(1)}(x) = \int_{-\infty}^{\infty} dy\ s_I^{(2)}(x, y) \quad (38)$$

A similar theorem can be formulated for the reduction from three to two dimensions.

By applying Fourier transformation to (38) with respect to x, and comparing the right-hand member with the Fourier transformation (3) for $S_I^{(2)}$, it is easily verified that

$$S_I^{(1)}(\lambda) = S_I^{(2)}(\lambda, 0)$$

Similarly

$$S_I^{(2)}(\lambda, \mu) = S_I^{(3)}(\lambda, \mu, 0)$$

This is also true in anisotropic fields.

In isotropic fields the above dimensional reduction involves that $S_I^{(m)}(\sigma)$ represents the same function for all values of m, provided that σ represents the polar radius of the spectrum space in all three cases.

6. *Neighbor diffusion.* On the Eulerian plan the neighbor concentration in one dimension is defined as an autocorrelation as follows:

$$n(t, \xi, p) = \int_{-\infty}^{\infty} dx\, s(t, x, p)\, s(t, x - \xi, p) \quad (39)$$

Let N be the Fourier transform of n, when λ is associated with ξ. Then it follows from (39) that

$$N(t, \lambda, p) = S(t, \lambda, p)\ S^*(t, \lambda, p) = |S(t, \lambda, p)|^2 \quad (40)$$

Here S^* is complex conjugated to S.

As *Ichiye* [1950] has shown, (40) indicates that the neighbor distribution only defines the amplitude spectrum of the ordinary local distribution, but not the phase spectrum, so that s cannot be uniquely defined from n.

When we deal with average distributions, another question arises in the translation from neighbor to local concentration:

Let $\bar{S}$ denote the statistical average of S and

$\bar{N}$ that of N. Let $\tilde{S}$ denote the random variation of S:

$$\tilde{S}(t, \lambda, p) = S(t, \lambda, p) - \bar{S}(t, \lambda)$$

Then it follows from (40) that

$$\bar{N}(t, \lambda) = |\bar{S}(t, \lambda)|^2 + \overline{|\tilde{S}(t, \lambda, p)|^2} \qquad (41)$$

Hence we cannot translate mean neighbor into mean local concentration without evaluating the random fluctuations in the local distribution.

If we suppose that the random variations are small with respect to the mean distribution, so that we may neglect the random contribution to (41) as a second-order effect, we have

$$\bar{N}(t, \lambda) \approx |\bar{S}(t, \lambda)|^2 \qquad (42)$$

Hereafter we shall use (42), omitting once more the bars denoting mean values.

Richardson [1926] postulated a neighbor diffusion equation of the form

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial n}{\partial \xi}\right] = 0 \qquad (43)$$

where the neighbor diffusivity a is supposed to depend on the neighbor separation parameter ξ.

The Fourier transform of (43) is

$$\frac{\partial N}{\partial t} + 2\pi\lambda \int_{-\infty}^{\infty} d\lambda_1$$

$$\cdot A(\lambda_1)\, 2\pi(\lambda - \lambda_1)\, N(t, \lambda - \lambda_1) = 0 \qquad (44)$$

where $A(\lambda)$ denotes the Fourier transform of $a(\xi)$.

In (43) the neighbor diffusion term is derived from the instantaneous neighbor distribution, and so we may criticize (43) as we criticized the derivation of (10).

Ignoring retardation effects in the integral diffusion as well as in the neighbor diffusion, we have in one dimension

$$(\partial S/\partial t) + 4\pi^2\lambda^2 K(\lambda) S = Q$$

from which

$$\frac{\partial N}{\partial t} + 8\pi^2\lambda^2 K(\lambda) N = QS^* + Q^*S \qquad (45)$$

can be deduced by using the relation (41). Here $Q = 0$ has to be put in order to make (45) comparable to (44).

Equations 45 and 43 are not compatible unless K is assumed to be constant and A is assumed to be

$$A = 2K\, \delta(\lambda_1)$$

which involves that $a = 2K$ is constant likewise.

Hence the assumption that the neighbor diffusivity a depends on the separation ξ does not seem compatible with the postulate of stationary, homogeneous turbulence.

7. *Some particular distributions.* Unit release distributions s_I have been deduced by several authors. We shall discuss this against the background of the preceding theory.

When

$$F(f') = 1 \qquad (46)$$

the retardation function becomes

$$F(t') = \delta(t')$$

which means that retardation is absent. Then (31) reduces to

$$\partial S/\partial t + 2\pi W\sigma S = Q \qquad (47)$$

which is essentially the same as (10).

After substituting (46) into (33) we obtain

$$G = e^{-t'}\epsilon(t')$$

where $\epsilon(t')$ denotes Heaviside's unit step function. Hence

$$S_I = e^{-2\pi W\sigma t}\epsilon(t) \qquad (48)$$

This can also be deduced from (47) [*Schönfeld,* 1959].

We arrive at the same equation 48 for one- and three-dimensional diffusion.

When $\alpha = 0$, so that $c = W =$ constant, (36) can be solved analytically, as well as the corresponding equations for one and three dimensions. This yields

$$H(r') = \frac{1}{c_m}[1 + (r')^2]^{-(m+1)/2}$$

and hence

$$s_I = \frac{1}{c_m}\frac{Wt}{\sqrt{W^2t^2 + r^2}^{m+1}}\,\epsilon(t) \qquad (49)$$

where $c_m = \pi$, 2π, or π^2 for $m = 1$, 2, or 3 dimensions, respectively.

When the concentration of diffusate has a normal distribution, we can satisfy (35) and the corresponding equations for $m = 1$ or 3 by postulating

$$s_I = b^{-m} e^{-\pi(r/b)^2} \epsilon(t) \qquad (50)$$

where

$$b = \frac{1}{\sqrt{\pi}} \left(\frac{1+\alpha}{2} \,!\, 2\pi c t \right)^{1/(1+\alpha)}$$

so that

$$S_I = e^{-\pi(b\sigma)^2}$$

for all m.

When $\alpha = 0$ (Okube, cf. *Pritchard and Carpenter* [1960]), (34) yields

$$G(t') = e^{-(\pi/4)(t')^2} \epsilon(t)$$

Then we can define F by inversion of (33), and finally F by transforming F. The last step has been done by numerical integration; the resulting retardation function is represented in Figure 1.

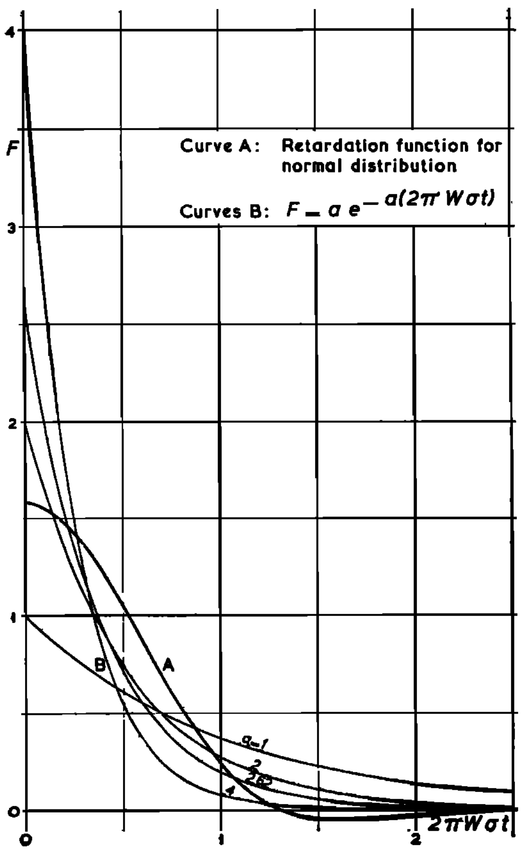

Fig. 1. Retardation functions.

A third type of distribution is of the form

$$s_I = \pi^{-m/2} \frac{(m/2)!}{[m/(1-\alpha)]!} \{(1-\alpha)^2 c t\}^{-m/(1-\alpha)} \cdot \exp^{-(r^{1-\alpha}/\{(1-\alpha)^2 c t\})} \epsilon(t) \qquad (51)$$

This was derived by *Joseph and Sendner* [1958] along a line of argument leading to an equation

$$\frac{\partial s}{\partial t} = r^{1-m} \frac{\partial}{\partial r} \left\{ c r^{\alpha+m} \frac{\partial s}{\partial r} \right\}$$

A simplified derivation for $\alpha = 0$ was given by *Bourret* [1959].

Previously *MacEwen* [1950] had discussed the case $m = 2$, $\alpha = 0$, of which Joseph and Sendner have given further applications.

The case $m = 2$, $\alpha = 1/3$ was treated by *Ozmidow* [1958].

The Fourier transform of (51) with respect to r can be deduced analytically when $\alpha = 0$:

$$S_I = [1 + (2\pi c t \sigma)^2]^{-(m+1)/2} \epsilon(t)$$

As this formula is not independent of m, the distribution (51) is incompatible with the dimensional reduction theorem (38).

The unretarded distribution (49) and the normal distribution (50) have been represented in Figure 2.

In order to investigate the retarded diffusion in more detail, we consider a retardation function of the form

$$F(t') = a\, e^{-at'} \epsilon(t') \qquad (52)$$

Then analytical integration is possible when $\alpha = 0$. Let

$$p = \sqrt{1 - (a/4)}$$

then

$$S_I = \left[\frac{1+p}{2p} e^{\{-2/(1+p)\} 2\pi W \sigma t} - \frac{1-p}{2p} e^{\{-2/(1-p)\} 2\pi W \sigma t} \right] \epsilon(t)$$

Hence for $m = 2$:

$$s_I = \frac{Wt}{2\pi p} \left[\left\{ \left(\frac{2Wt}{1+p} \right)^2 + r^2 \right\}^{-3/2} - \left\{ \left(\frac{2Wt}{1-p} \right)^2 + r^2 \right\}^{-3/2} \right] \epsilon(t) \qquad (53)$$

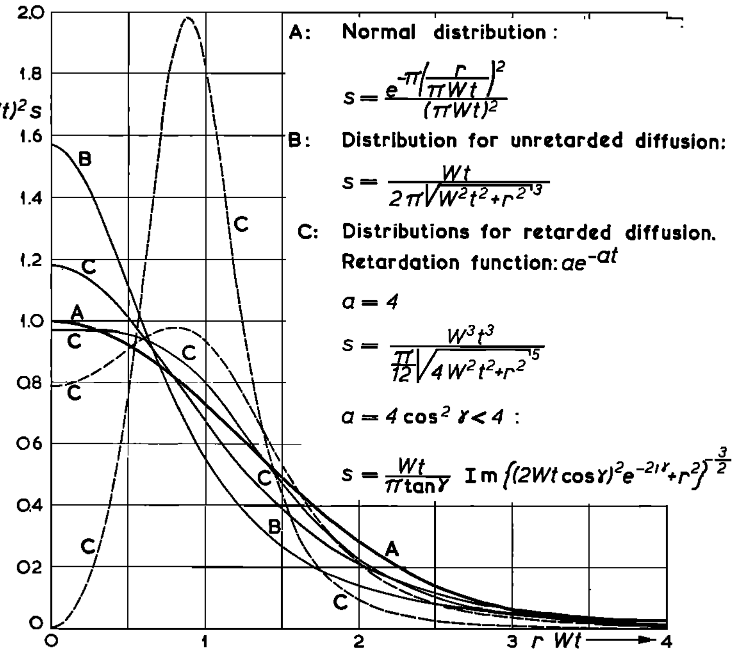

Fig. 2. Unit release distributions.

For $a = \infty$, we have $p = 1$ and (53) reduces to (49).

When $a = 4$, we have $p = 0$ and (52) can be reduced to

$$s_I = \frac{12}{\pi} \frac{(Wt)^3}{\sqrt{4(Wt)^2 + r^2}^5} \tag{54}$$

with the aid of a limit procedure.

When $a < 4$, p becomes imaginary and (53) can be reduced to

$$s_I = \frac{Wt}{\pi \tan \gamma}$$

$$\cdot \mathrm{Im}\ \{(2Wt \cos \gamma)^2 e^{-2i\gamma} + r^2\}^{-3/2} \epsilon(t) \tag{55}$$

if we put $p = i \tan \gamma$. The operator Im defines the imaginary part of the following quantity.

In Figure 1 the retardation function (52) has been represented for: $a = 4$ $[\gamma = 0]$; $a = \frac{1}{2}(3 + \sqrt{5})$ $[\gamma = \pi/5]$; $a = 2$ $[\gamma = \pi/4]$; $a = 1$ $[\gamma = \pi/3]$. The corresponding distributions appear in Figure 2.

When a decreases, the retardation becomes more prolonged. As illustrated by Figure 2, the distribution first becomes more uniform in the center and steeper at the periphery when a decreases. When $a < \frac{1}{2}(3 + \sqrt{5})$, the concentration in the center is depressed between that of the vicinity. For $a < 1$ the concentration in the center is negative.

It is clear that distributions as for $a < 1$ are physically impossible, and distributions as for $1 < a < \frac{1}{2}(3 + \sqrt{5})$ unrealistic. They can be explained formally as follows:

Immediately after the release, there is a strong divergent diffusive transport near the origin, and very little transport at greater distances. The retardation tends to prolong this pattern of transport also when the cloud is further dispersed. Hence, when the delay is great, the cloud may be hollowed out at the center.

These examples demonstrate that there can be retardation to only a limited degree.

8. *Empirical evidence.* Turbulence in the oceans and their adjacent waters is due to a variety of causes, and the type of turbulence may therefore be different from one situation to another. There are situations, for instance, in which the turbulence clearly appears nonstationary, or inhomogeneous, or anisotropic, or a combination of these possibilities. Nevertheless, to some extent it seems justified to adopt the concept of stationary, homogeneous, and isotropic turbulence as a basis of description

of oceanic turbulence, treating the effects of nonstationarity, etc., as secondary and capable of elimination by averaging processes. When we consider larger scales of time and length, for instance, we may expect that nonstationarity and inhomogeneity on smaller scales are smoothed out in the average. Similarly, anisotropy will be smoothed out by averaging by different directions.

When stationary, homogeneous, and isotropic turbulence obeys a law of similarity as formulated in section 5, two functions appear to be sufficient to characterize the diffusive properties of the field. The first function, approximated by a power expression, defines the correlation between length and time scales of the eddies. The second function defines the retardation of the diffusive effect.

The decay of a unit release distribution (35) is determined by the power law (27). The shape of such a distribution follows from the retardation function. Hence both characteristic functions can be deduced by recording a unit release distribution both in time and in space.

A release experiment partly covering these requirements has been described by *Folsom and Vine* [1957]. Since the cloud was only recorded an instant after the release, it is not possible to deduce the correlation exponent from this experiment.

Other observations in the ocean seem to indicate a value of 0 to $1/3$ for the exponent α.

The value $\alpha = 1/3$, valid for the Kolmogoroff range, has been ascertained for smaller-scale turbulence (say a few kilometers at most) by several authors [*Ozmidow*, 1958, 1959; *Ichiye and Olson* [1960]. On the other hand, a smaller value of α seems more probable for larger scales [*Joseph and Sendner*, 1958]. *Pritchard and Carpenter* [1960] even claim $\alpha = 0$ for rather small scales.

A diffusion process on a very large scale is observed in the spreading of the Mediterranean Sea water in the Atlantic Ocean. By plotting the salinity against the area of the isohaline curves, Joseph and Sendner obtained a picture from which possible anisotropy has been eliminated. As the distribution of the Mediterranean Sea water is steady, broadly speaking, nonstationarities in the turbulent field may also be supposed to have been eliminated. For lack of a better assumption, homogeneity is also postulated. Advection by the flow expelled from the Strait of Gibraltar must be accounted for.

As Joseph and Sendner's theory is not entirely satisfactory, we have recomputed the

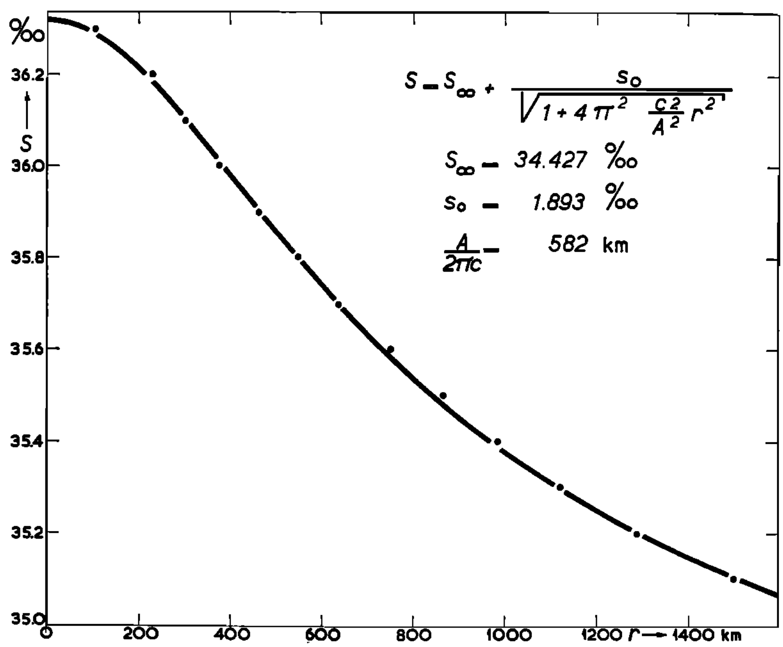

Fig. 3. Diffusion of the Mediterranean Sea water in the Atlantic Ocean.

spreading of the Mediterranean Sea water on the basis of the theory here advanced.

When we assume, with Joseph and Sendner, and for lack of a better assumption, that the Mediterranean Sea water is absorbed by the surrounding waters at great distances from the origin, we obtain the following equation in the $t\sigma$ plane [*Schönfeld,* 1959]:

$$2\pi c\sigma^{1-\alpha} S(\sigma) + 2\pi A \int_0^{\sigma} S(\sigma_1)\sigma_1 \, d\sigma_1 = As_0 \qquad (56)$$

Here A represents the flow of water from the origin, s the excess of the salinity of this water over that of the surrounding water, and s_0 the original value of this excess.

The solution of (56) is

$$S = \frac{s_0 A}{2\pi c} \sigma^{\alpha-1} \cdot \exp^{-[(A/c)(\sigma^{1+\alpha}/[1+\alpha])]} \qquad (57)$$

For $\alpha = 0$ this can be transformed analytically into

$$s = \frac{s_0}{\sqrt{1 + 4\pi^2(c^2/A^2)r^2}} \qquad (58)$$

which is in very good agreement with the observations if the salinity of the surroundings, the original salinity of the Mediterranean Sea water, and the parameter A/c are suitably adapted (see Fig. 3).

When $\alpha = 1/3$, the inverse transformation of (57) requires numerical integration. We are engaged in this, but no result is available. Estimating from the function (57) we expect that $\alpha = 1/3$ will not fit as well as $\alpha = 0$.

For a better evaluation of this question, however, the possible mixing of the Mediterranean Sea water with the overlying Atlantic Ocean water should also be taken into consideration; this is possible with numerical integration.

When we adopt $\alpha = 0$, we can interpret the experiment of Folsom and Vine, for instance, by (49), (50), or (54). This has been represented in Figure 4.

The unretarded distribution (49) seems to fit better than the normal distribution (50). The somewhat intermediate distribution (52) yields a still better fit. This would indicate that there is some retardation but less than should be assumed to explain a normal distribution (see Fig. 1).

This conclusion is in agreement with the interpretation of *Pritchard and Carpenter* [1960].

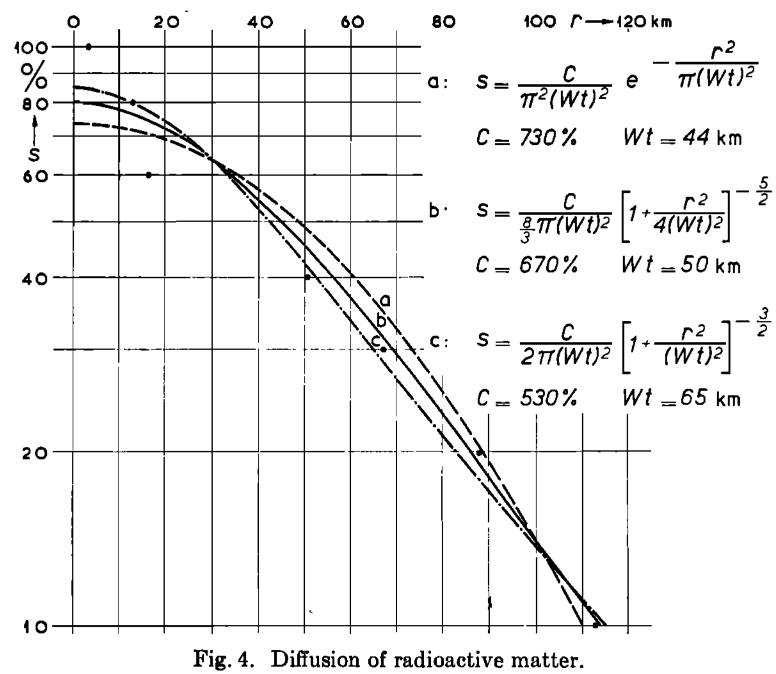

Fig. 4. Diffusion of radioactive matter.

REFERENCES

Bourret, R., Derivation of the Joseph-Sendner diffusion equation from the Jaynes' principle of statistical inference, *Deut. Hydrograph. Z., 12*(4), 180–181, 1959.

Folsom, T. R., and A. C. Vine, On the tagging of water masses for the study of physical processes in the oceans, *Woods Hole Oceanog. Inst. Contrib. 929*, pp. 121–132, 1957.

Frenkiel, F. N., Mean concentration distribution in a flow field of homogeneous turbulence, *Advances in Appl. Mech., 3*, 61–107, 1953.

Groen, P., On the behaviour of gravity waves in a turbulent medium, with application to the decay and apparent period increase of swell. *K.N.M.I. commun.*, no. 63, 23 pp., 1954.

Ichiye, T., Some remarks on Richardson's neighbour diffusion equation. *Oceanog. Mag. Tokyo, 2*(3), 101–104, 1950.

Ichiye, T., and F. C. W. Olson, Ueber die neighbour diffusivity im Ozean, *Deut. Hydrograph. Z., 13*(1), 13–23, 1960.

Joseph, J., and H. Sendner, Ueber die horizontale Diffusion im Meere, *Deut. Hydrograph. Z., 11*(2), 49–77, 1958.

Kolmogoroff, A., Ueber die analytischen Methoden in der Wahrscheinlichkeitsrechnung, *Math. Ann., 104*, 415–458, 1931.

MacEwen, G. F., A statistical model of instantaneous point and disc sources with application to oceanographic observations, *Trans. Am. Geophys. Union, 31*(1), 35–46, 1950.

Ozmidow, R. V., Horizontal turbulent diffusion in the sea, *Akad. Nauk USSR, 120*(4), 761–764, 1958.

Ozmidow, R. V., Horizontal turbulence in the ocean, *Akad. Nauk SSSR, 126*(1), 63–65, 1959.

Pritchard, D. W., and J. H. Carpenter, Measurements of turbulent diffusion in estuarine waters, Report presented at IUGG meeting, Helsinki, 13 pp., 1960.

Richardson, L. F., Atmospheric diffusion shown on a distance-neighbour graph, *Proc. Roy. Soc. London, A, 110*, 709–737, 1926.

Schönfeld, J. C., Diffusion by homogeneous, isotropic turbulence, *Rept. Rijkswaterstaat*, 31 pp., 1959.

Stommel, H., Horizontal diffusion due to oceanic turbulence, *J. Marine Research (Sears Foundation), 8*(3), 199–225, 1949.