On the Spectrum of the Mean Diffusion Velocities in the Ocean[1]

J. JOSEPH AND H. SENDNER

Deutsches Hydrographisches Institut, Hamburg, Germany

Abstract. By interpretation of the concentration s of a diffusing substance in the sea as probability density, a Fokker-Planck differential equation of the second order is derived from a Markov functional equation for s. For small time differences only the momenta of distance of the first and second order are taken into account. Scales of length r are determined as radii of the circles that enclose the areas bounded by isoconcentration lines. The area mean of the diffusion velocity, given by the momentum of the first order, yields a velocity of $2P(r)$. These results are compared with experimental observations, and they indicate a dependence on r similar to that assumed by Stommel for the velocity spectrum in the ocean.

For some time now diffusion processes have been of major interest to oceanographers, particularly since radioactive wastes are being discharged into the sea. At present the question of the dispersion of these wastes in different sea areas cannot be answered satisfactorily, because any prediction calls for knowledge of a law capable of being extrapolated for diffusion processes. The turbulence processes are too complicated to be reliably computed on the basis of theoretical considerations. Our aim, therefore, should be to interpret the observations theoretically in such a way that a prediction is possible within certain limitations. We have tried to apply this to a very special problem, namely, the large-scale horizontal diffusion of a substance which initially is concentrated at a point. The first question raised was how observations on diffusion could be expressed so that they could be theoretically interpreted. We discussed this problem in one of our earlier articles [*Joseph and Sendner*, 1958]. We shall consider areas enclosed by lines of constant concentration of the diffusing substance.

Figure 1 illustrates a possible pattern of the spreading of such a substance 1 and 2 days after its release. The maximum concentration reached after the first day at a point in this area

is set as 100. As a measure of the area enclosed by an isoconcentration curve we have defined the radius r of a circle of equal area. If the distribution of the concentration is now regarded as a function of time and of this equivalent radius r, the observations will show that this function is roughly expressed by the equation

$$s(r, t) = \frac{M}{2\pi(Pt)^2} e^{-r/Pt} \tag{1}$$

where s is the concentration, r the linear measure of the area, t is the time, and M the quantity of the diffusing substance. P is a term having the dimension of a velocity.

If $s(R; r, t)$ is the probability density that after a time t one particle travels from distance R to distance r, then (see *Joseph and Sendner* [1958]) the functional equation by Markov is applicable
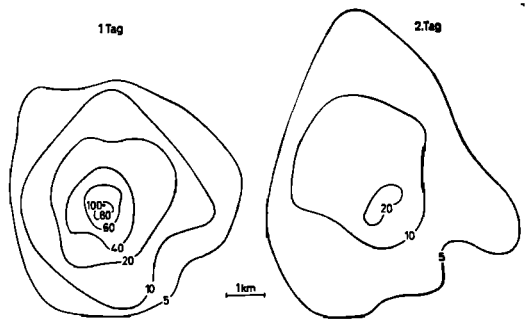
Fig. 1. Distribution of the concentration of a diffusing substance after 1 day (left) and after 2 days (right).

TABLE 1

S, ‰	Area, km²	r, km
10	40,000	112.8
20	24,000	87.4
30	14,000	66.8
40	8,000	50.5
60	800	16.0
80	490	12.5
100	35	3.3

$$s(R; r, t + \Delta t)$$

$$= \int_0^\infty s(R; \rho, t) s(\rho; r, \Delta t) 2\pi\rho \, d\rho \qquad (2)$$

If for the momenta of first and second order the mean diffusion velocity $2P$ is taken as constant and the mean square deviation is taken as proportional to the linear measure r (such as $2Pr$), then

$$\frac{\partial s}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r}\left(r^2 \cdot P \frac{\partial s}{\partial r}\right) \qquad (3)$$

provided that the diffusing substance M is concentrated at one point at time $t = 0$.

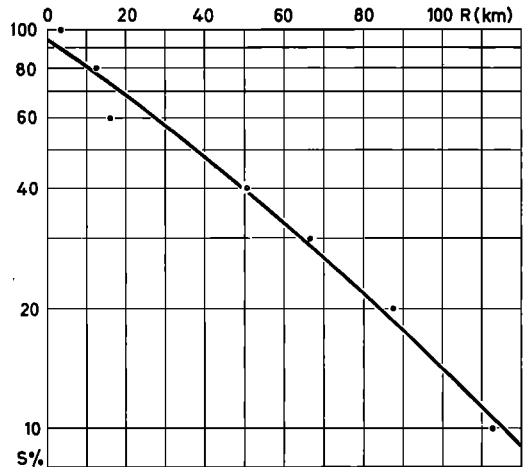It appeared that the observations could be represented even better with a variable $P = P(r)$. Therefore we write

Fig. 2. Diffusion diagram of radioactive test material in the ocean.

$$\lim_{\Delta t \to 0} \int_0^\infty \frac{\rho - r}{\Delta t} s(r; \rho, \Delta t) 2\pi\rho \, d\rho$$

$$= \frac{1}{r} \frac{\partial}{\partial r} (r^2 P(r)) = \bar{v}(r) \qquad (4)$$

$$\lim_{\Delta t \to 0} \int_0^\infty \frac{(\rho - r)^2}{\Delta t} s(r; \rho, \Delta t) 2\pi\rho \, d\rho$$
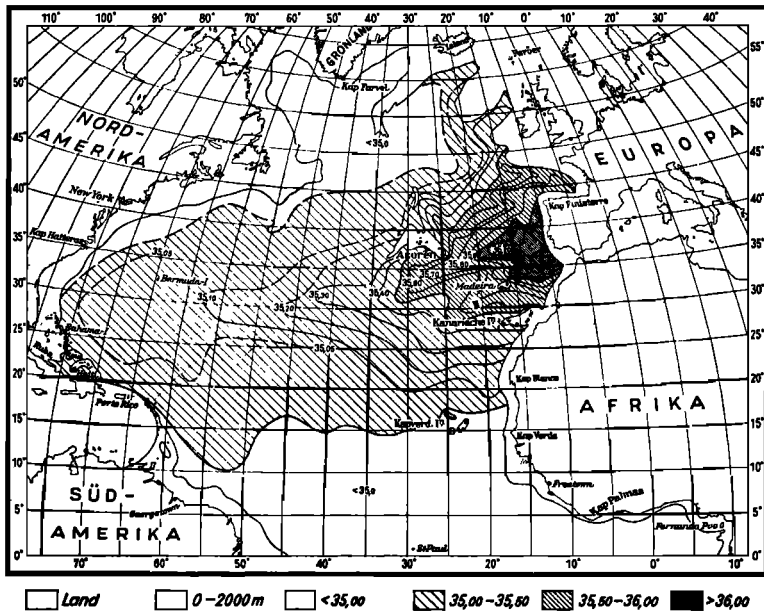
$$= 2rP(r)$$

Land   0-2000m   <35,00   35,00-35,50   35,50-36,00   >36,00

Fig. 3. Spreading of Mediterranean Sea water into the Atlantic Ocean according to *Wüst* [1935].

TABLE 2

S Observed, ‰	Area, 10^3 km²	r, km	S Computed, ‰	ΔS, ‰
36.30	34.6	105.0	36.300	0.000
36.20	167.0	230.6	36.190	+0.010
36.10	278.4	297.7	36.106	−0.006
36.00	441.3	374.8	36.008	−0.008
35.90	668.4	461.3	35.901	−0.001
35.80	951.5	550.3	35.797	+0.003
35.70	1275.1	637.1	35.704	−0.004
35.60	1771.4	750.9	35.593	+0.007
35.50	2359.8	866.7	35.493	+0.007
35.40	3045.7	984.6	35.401	−0.001
35.30	3956.4	1122.2	35.306	−0.006
35.20	5222.6	1289.4	35.205	−0.005
35.10	7062.2	1499.3	35.096	+0.004

where $\bar{v}(r)$ corresponds with the mean diffusion velocity. Higher momenta are neglected. Thus, according to *Kolmogoroff* [1931] equation 3 with $P = P(r)$ is obtained. For the integration a power law in the following form was considered for $P(r)$

$$P(r) = P_0(r/r_0)^{\lambda-1} \qquad (5)$$

$\lambda = 1$ corresponds to the event $P = $ constant. If again we assume that at time $t = 0$ the substance is concentrated at the point $r = 0$, then instead of (1) we find

$$s(r,\ t) = \frac{2-\lambda}{2\pi r_0^2}\ \frac{M}{\Gamma\left(\dfrac{2}{2-\lambda}\right)}$$
$$\cdot\frac{\exp\left[-\dfrac{r_0}{(2-\lambda)^2 P_0 t}\left(\dfrac{r}{r_0}\right)^{2-\lambda}\right]}{\left[(2-\lambda)^2\ \dfrac{P_0 t}{r_0}\right]^{2/(2-\lambda)}} \qquad (6)$$

Let us test this equation by means of the observations. Unfortunately large-scale oceanic observations of diffusion processes are too few to permit us to draw lines of equal concentration with sufficient accuracy. The only observations available for a nonstationary process are those made by *Folsom and Vine* [1957], who studied concentrations with radioactive material. Within a small area which can approximately be regarded as a point, such material was released onto the surface of the sea and its distribution measured 40 days afterward. From these observations we have estimated, by our methods, the radii of the areas (Table 1) covering a region ranging from approximately 10 to 100 km.

Figure 2 presents the computed radii and function $s(r,\ t)$ according to (6). Employing the method of least squares, we find for P_0 1.6 cm/sec with r_0 of 12.5 km. In this example the power law (5) will be

$$P = P_0(r/r_0)^{-0.17}$$

In view of the small exponent 0.17 the dimension $P(r)$ is not very variable in the observed interval.

The classical example for stationary large-scale mixing is the spreading of Mediterranean Sea water into the Atlantic Ocean as described by *Wüst* [1935] and represented in Figure 3. On account of diffusion processes, water of high salinity stretches from the Strait of Gibraltar down to a depth of about 1000 to 1100 meters, its salinity being gradually reduced by mixing with neighboring water masses. As the distribution of the core layer takes place in an isentropic layer, the problem can be treated two-dimensionally. When applying the diffusion equation to this problem we have to consider the inflow of about 50,000 km³ of Mediterranean Sea water per year from the Strait of Gibraltar. This is done by adding, to the mean diffusion velocity $\bar{v}(r)$ in our first postulate (4), the mean velocity of the amount of water that originates from the source $v_0 r_0/r$ (where $2\pi r_0\ v_0$ is the constant efficiency of the source). Likewise the following differential equation results for s:

$$\frac{\partial s}{\partial t} = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r^2\cdot P(r)\ \frac{\partial s}{\partial r}\right) - \frac{r_0 v_0}{r}\cdot\frac{\partial s}{\partial r} \qquad (7)$$
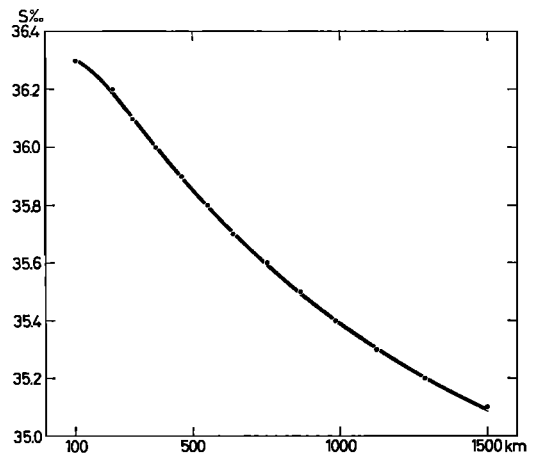

Fig. 4.   Diffusion diagram of Mediterranean Sea water in the Atlantic Ocean.

JOSEPH AND SENDNER

and as solution for the stationary case $\partial s / \partial t = 0$ with the power theorem (5) for $P(r)$

$$s(r) = S_0 + S_1 \cdot \exp \left[-(v_0/\lambda P_0)(r_0/r)^\lambda \right] \quad (8)$$

The integration constants S_0 and S_1 have to be determined by the initial and final salinity.

In Table 2 are compiled the areas (10^3 km²) within the isohalines (S ‰) and the distances (r km) computed therefrom. The constants v_0/P_0, λ, S_0, and S_1 were determined by adjustments. The following power law is obtained:

$$P(r) = P_0(r/r_0)^{-0.30}$$

For $r = 105$ km there results $v_0/P_0 = 3.83$. If, with Defant, $v_0 \approx 4$ cm/sec, then $P_0 \approx 1$ cm/sec. The discrepancy between observation and computation is given in the last column of Table 2. The mean error amounts to ± 0.007 S‰ and lies below the titration accuracy, which means that the observations are represented by equation 8 in a region whose area has a radius between 100 and 1500 km. The mixing diagram thus obtained is given in Figure 4, in which the observed values and the estimated salinity are given as functions of the radii of the areas.

These two examples show that λ of the power law (5) can be constant only within a certain region. Thus the question arises in what way the diffusion velocity $2P(r)$ depends upon r. As the diffusion processes are a result of the turbulent mixing processes, a close connection between $P(r)$ and the mean turbulent velocities must be expected.

According to studies by *Kolmogoroff* [1941a, b, c] and *von Weizsäcker* [1948], the temporal mean value of the turbulent velocities is proportional to the third root of the linear dimension
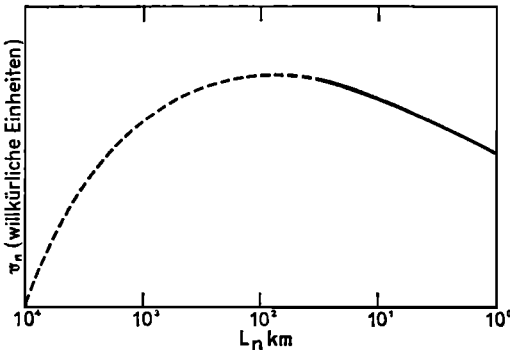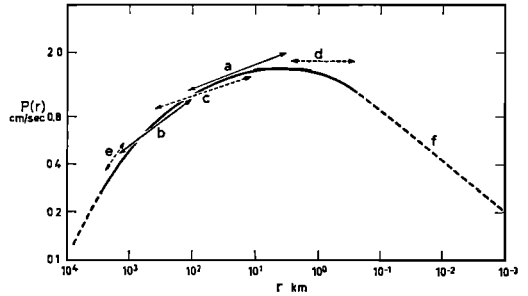

Fig. 6. Spectrum of diffusion velocities in the Atlantic Ocean.

in the observed region. On the basis of these considerations *Stommel* [1949] has set up a hypothetic turbulence spectrum for the ocean, in which the turbulence velocity is given in arbitrary units.

Figure 5 shows the sketch made by Stommel for the Atlantic Ocean. For the region to the right of the figure the 1/3 power law is valid. An analogous diffusion velocity spectrum is presented in Figure 6, using the few available observations. a and b are the power laws resulting from the two discussed examples; c, d, and e are based on less extensive observations of presumed diffusion processes. For c the power law was calculated from observations of the turbidity distribution in the Irminger See [*Joseph*, 1958], and for e from the frequency distribution of sargassum [*Krümmel*, 1892]. For d only the value for P = constant could be determined from observations of the diffusion of radioactive material in the Bikini atoll [*Munk, Ewing, and Revelle*, 1949]. f corresponds to the 1/3 power law.

REFERENCES

Folsom, T. R., and A. C. Vine, On the tagging of water masses for the study of physical processes in the oceans, *Contrib. 929, Woods Hole Oceanogr. Inst.*, 1957.

Joseph, J., Die Trübungsverhältnisse in der Irminger See im Juni 1955 und ihre hydrographischen Ursachen, *Ber. deut. wiss. Kommission für Meeresforschung*, N. F. *14*, 1958.

Joseph, J., and H. Sendner, Ueber die horizontale Diffusion im Meere, *Deut. Hydrograph. Z.*, *11*, 2, 1958.

Kolmogoroff, A. N., 1931, Ueber die analytischen Methoden der Wahrscheinlichkeitsrechnung, *Math. Ann.*, *104*, 415, 1931.

Kolmogoroff, A. N., Die lokale Struktur der Turbulenz in einer inkompressiblen zähen Flüssig-

Fig. 5. Turbulence spectrum for the ocean according to *Stommel* [1949].

keit bei sehr grossen reynoldschen Zahlen, *Doklady Akad. Nauk SSSR*, *30*(4), 1941*a*.

Kolmogoroff, A. N., Die Energiedissipation für lokalisotrope Turbulenz, *Doklady Akad. Nauk SSSR*, *32*(1), 19, 1941*b*.

Kolmogoroff, A. N., Zur Entartung der isotropen Turbulenz in einer inkompressiblen zähen Flüssigkeit, *Doklady Akad. Nauk SSSR*, *31*(6), 583, 1941*c*.

Krümmel, O., *Reisebeschreibung der Plankton-Expedition mit Auszug daraus*, Kiel and Leipzig, 1892.

Munk, W. H., G. C. Ewing, and R. R. Revelle, Diffusion in Bikini lagoon, *Trans. Am. Geophys. Union*, *30*, 59, 1949.

Stommel, H., Horizontal diffusion due to oceanic turbulence, *J. Marine Research*, *8*(1), 199, 1949.

von Weizsäcker, C. F., Das Spektrum der Turbulenz bei grossen reynoldschen Zahlen, *Z. Physik*, *124*, 614, 1948.

Wüst, G., Die Stratosphäre des atlantischen Ozeans, *Wiss. Ergeb. deut. atlantischen Expedition 'Meteor,' 1925–27*, 6 (text), part 1, no. 2, Berlin, 1935.