Diffusion in the Diabatic Surface Layer[1]

F. A. GIFFORD

U. S. Weather Bureau Research Station, Oak Ridge, Tennessee

Abstract. Diffusion from point and line sources in an adiabatic surface layer were briefly discussed using a similarity method by Ellison, following a suggestion made at the 1958 Oxford Symposium by Batchelor. The chief result is that the ground concentration at a downwind distance x, from a point source at ground level, is found to be proportional to x^m, where m lies in the range -1.85 to -1.9. Assuming that Lagrangian similarity of the surface layer turbulence obtains also in the diabatic case, the power index, m, of the downwind ground level concentration is shown, in the present work, to vary from about -1.35 in very stable conditions to -3.0 in superadiabatic conditions. This result is in good agreement with recent detailed concentration measurements.

1. *Introduction.* Following a suggestion by *Batchelor* [1959*a*], *Ellison* [1959] was able to apply the concept of Lagrangian similarity to the analysis of turbulent diffusion from a point source in a logarithmic, i.e. an adiabatic, or neutrally (indifferently) stratified atmospheric surface layer (the layer of a few tens of meters in depth just next to the ground through which the stress does not vary significantly). Lagrangian similarity involves the idea that the statistical properties of the motion of a fluid point, such for example as its mean position, may remain self-similar at all positions in a turbulent flow; that is, the mathematical functions describing these properties can differ only by a scale factor, from point to point in the flow. Lagrangian similarity had previously been discussed in connection with diffusion in jets and wakes by *Batchelor* [1957]. Ellison derived the equation for the relative concentration downwind from a continuous point source at ground level, as well as that for an infinite cross wind line source, and reported that Batchelor had also obtained these results in an unpublished note on the diffusion from sources in a turbulent boundary layer.

The well-known idea of Eulerian similarity of turbulent flows, applied with considerable success to the determination of the velocity and temperature profiles in the diabatic (i.e., thermally stratified, or nonneutral) surface layer by *Monin and Obukhov* [1954], has recently been used by *Kazansky and Monin* [1957] (see also *Monin* [1959]) to determine the shape of the boundary of smoke plumes within the surface layer. A related result was reported by *Kao* [1961], based on his similarity model of the diabatic surface layer; reference should also be made to *Inoue's* [1960] interesting study, in which a vertical diffusion coefficient was derived on the basis of similarity considerations.

The successful verification of Eulerian similarity theory over a fairly wide range of stability conditions, for example in the studies by *Taylor* [1960*a*, *b*]; *Priestley* [1955, 1959] *Panofsky, Blackadar, and McVehil* [1960]; and *Takeuchi* [1961], encourages the hypothesis that Lagrangian similarity is likewise a characteristic property of the diabatic atmospheric surface layer. If this assumption is made, it is possible to extend the diffusion results given by Ellison, and Batchelor, to the case of the diabatic surface layer, which is the purpose of this paper.

According to the hypothesis of Eulerian similarity, as applied to a turbulent atmospheric surface layer, the flow characteristics are all determined entirely by the friction velocity v_* and a length L such that

$$L = \frac{v_*^{\,3}}{k(g/T)(-q/c_p\rho)}$$

k is von Kármán's constant, equal to 0.40; g is the acceleration of gravity, T is average

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temperature, q is heat flux in the vertical direction, ρ is air density, and c_p is specific heat at constant pressure. L is positive for negative heat flux (stable conditions), and vice versa, and becomes infinite under adiabatic conditions. A similar characteristic length was defined by *Lettau* [1949]. Now if a particle is released at ground level and diffuses in the surface layer, it will reach some average position characterized by coordinates $\bar{x}(t)$, $\bar{y}(t)$, and $\bar{z}(t)$; the downwind direction is chosen to coincide with that of $\bar{x}$. The time t is counted from the time of release at the initial point of reference. The average vertical velocity of such a particle, $\bar{w}$, must on dimensional grounds be proportional to v_*, times some (as yet unspecified) universal, dimensionless function involving L. Therefore we can write

$$d\bar{z}/dt \equiv \bar{w} = bv_*\phi(\zeta) \tag{1}$$

where ϕ is a universal function of the dimensionless variable $\zeta = \bar{z}/L$, and b is a fundamental constant, equal to the ratio of the mean vertical velocity of a particle under adiabatic conditions to v_*. Since for adiabatic conditions $L = \infty$, it follows that $\phi(0) = 1$, according to these definitions. Similarly, the average horizontal velocity of a particle, $\bar{u}$, is given by

$$d\bar{x}/dt \equiv \bar{u} = (v_*/k)[f(\zeta) - f(\zeta_0)] \tag{2}$$

where $\zeta_0 \equiv z_0/L$; z_0 is the roughness length. The new universal function f has been written in this particular way, and the fundamental constant has been identified as k^{-1}, because it seems reasonable to suppose that the mean velocity of the diffusing particle at a point will coincide with the mean wind velocity there. The universal functions f and ϕ are for this reason expected to be identical with those given by *Monin and Obukhov* [1954] and *Kazansky and Monin* [1957] in their discussions of Eulerian similarity. In general, such functions must of course be determined by recourse to experiment.

Application of the similarity idea to diffusion from a point source now proceeds as follows. The probability ψ that a particle diffusing in the surface layer will reach a distance $r = (x^2 + y^2 + z^2)^{1/2}$ from its average position must be a universal function of $(x - \bar{x})/\bar{z}$, $y/\bar{z}$ and $(z - \bar{z})/\bar{z}$. On dimensional grounds, the average concentration of particles released instantaneously from the coordinate origin, X_{ips}, must then be

$$X_{ips}(x, y, z) = \frac{Q_i}{\bar{z}^3}\,\psi\left(\frac{x-\bar{x}}{\bar{z}}, \frac{y}{\bar{z}}, \frac{z-\bar{z}}{\bar{z}}\right) \tag{3}$$

where Q_i is the instantaneous point source strength. For a continuous source, the average concentration is found by a time integration in the usual manner:

$$X_{cps} = Q\int_0^\infty \psi\left(\frac{x-\bar{x}}{\bar{z}}, \frac{y}{\bar{z}}, \frac{z-\bar{z}}{\bar{z}}\right)\frac{dt}{\bar{z}^3} \tag{4}$$

where Q is the continuous point source strength. From equation 1 it follows that

$$X_{cps}/Qbv_* = \int_0^\infty \frac{\psi\left(\dfrac{x-\bar{x}}{\bar{z}}, \dfrac{y}{\bar{z}}, \dfrac{z-\bar{z}}{\bar{z}}\right)}{\phi(\bar{z}/L)}\frac{d\bar{z}}{\bar{z}^3} \tag{5}$$

Introducing a change of the integration variable, and restricting the problem to consideration of downwind ground level concentrations ($y = z = 0$), we find that

$$\frac{X_{\text{axial}}}{Q}\,bv_* = \int_0^\infty \frac{\psi\left(\dfrac{x-\bar{x}}{\bar{z}}, 0, 1\right)}{\phi\left(\dfrac{\bar{z}}{L}\right)\bar{z}^2}\,\frac{d\left(\dfrac{x-\bar{x}}{\bar{z}}\right)}{\left[\dfrac{x-\bar{x}}{\bar{z}} + \dfrac{d\bar{x}}{d\bar{z}}\right]} \tag{6}$$

This expression can be simplified, in the same way that homogeneous plume diffusion models usually are [*Frenkiel*, 1953], by neglecting diffusion in the x direction compared with that in the y and z directions. As was pointed out by *Batchelor* [1959*b*] and by *Ellison* [1959], this amounts to assuming that the plume element (the cloud of probability density that can be imagined to be attached to the mean position of the diffusing particle) is carried past any downwind point $\bar{x}$ in a time short compared with the time required to reach that point from the origin of diffusion. Consequently, $x \simeq \bar{x}$, and the only contribution to the integral in (6) comes from values of the argument near zero. Introducing (1) and (2), it then develops that

$$\frac{X_{\text{axial}}}{Q} \propto \frac{k}{\bar{z}^2 v_*[f(\zeta) - f(\zeta_0)]} = \frac{1}{\bar{z}^2\bar{u}(\bar{z}/L)} \tag{7}$$

In the adiabatic case, the axial concentration is then obtained by introducing the logarithmic form of equation 2, for the velocity profile, into equation 7. In the more general case, it is necessary to consider appropriate diabatic forms for $\bar{u}$. These are now comparatively well known, as a result of the various observational studies to which references were previously made. The conclusion from these studies is that surface-layer turbulence is characterized by three regimes: stable, forced convection, and free convection. The position seems to be that the velocity profile for the stable and forced convection cases is well represented by the log-plus-linear law of Monin and Obukhov,

$$f(\zeta) = \ln \zeta + \alpha\zeta \tag{8}$$

at least in a stability range not too different from neutral. The studies by Taylor, and Takeuchi, imply that this equation may be refined somewhat by adjusting the constant α over various ζ ranges. In the free convection case, Kazansky and Monin derived the interpolation formula

$$f(\zeta) = \begin{cases} \ln \zeta & 0 \geq \zeta \geq \zeta^* \\ c(\zeta^{-1/3} - \zeta^{*-1/3}) & \zeta < \zeta^* \end{cases} \tag{9}$$

on dimensional grounds.

Together with equation 7 these provide a relation between concentration and $\bar{z}$ that may be converted into a relationship with downwind distance $\bar{x}$ by introducing the combination of equations 1 and 2:

$$\frac{d\bar{x}}{d\bar{z}} = \frac{v_*}{k}\,[f(\zeta) - f(\zeta_0)]\cdot\frac{1}{bv_*\phi(\zeta)} \tag{10}$$

The function ϕ has been shown by *Monin* [1959], from the equation of energy balance, to be related to the function f by

$$\phi = \left[1 - \frac{1}{f'(\zeta)}\right]^{1/4} \tag{11}$$

where it is of interest that $1/f'(\zeta)$ is proportional to the Richardson number. It can be seen that the integral of (10) is

$$F(\zeta, \zeta_0) \equiv \xi$$

$$= \frac{bk\bar{x}}{L} = \int_{\zeta_0}^{\zeta} \frac{[f(\zeta) - f(\zeta_0)]}{\{1 - [1/f'(\zeta)]\}^{1/4}}\,d\zeta \tag{12}$$

Taken in combination with equations 7, 8, and 9, equation 12 defines the downwind ground concentration from a continuous point source located at the coordinate origin in the surface layer. The downwind axial concentration equations obtained by *Batchelor* [1959*b*] and *Ellison* [1959] for a logarithmic surface layer follow directly upon substitution into these equations of the logarithmic velocity profile, remembering that for this case $\phi = 1$. The result for a continuous point source is

$$\frac{X_{\text{axial}}}{Q}\,bv_* \propto \frac{1}{\bar{z}^2[(\bar{x}/\bar{z}) + (1/kb)]} \tag{13}$$

where

$$\bar{x} = \frac{1}{kb}\left[\bar{z}\left(\ln\frac{\bar{z}}{z_0} - 1\right)\right] \tag{14}$$

2. *Evaluation of downwind ground concentration in the diabatic case.* It is necessary, in order to determine concentration as a function of $\bar{x}$ in the diabatic case, to evaluate the functions f, ϕ, and F, defined by equations 8, 9, 11, and 12. This in turn requires that the constants α, c, and ζ^* also be determined. The constant α was originally assigned the value 0.6 by *Monin and Obukhov* [1954] on the basis of experimentally determined wind profiles; but, as a result of studies by *Priestley* [1959], *Taylor* [1960*a*], *Inoue* [1959], *Takeuchi* [1961], and *Panofsky, Blackadar, and MacVehil* [1960], it appears that α should be somewhat larger; the value $\alpha = 6$ is consistent with results given by most of these authors. *Taylor* [1960*a*, *b*] determined from observations that $c \approx 1.3$, a value close to that ($c = 1$) assumed by *Kazansky and Monin* [1957]. Taylor likewise found that the free convection form of the velocity profile, equation 9, applies for $\zeta < -0.08$, from which it is concluded that $\zeta^* = -0.08$. Although some residual uncertainty attaches to these suggested values, owing largely to the fact that the experimental wind and temperature profiles upon which they depend are not yet perhaps as numerous or as well established as we might wish, it is unlikely that they are seriously in error.

The function F was given in graphical form by *Monin* [1959], on the basis of a numerical integration that employed equations 8, 9, and 11 and the constant values given by *Kazansky and Monin* [1957]. For negative values of ζ, that is, for unstable conditions, it appears that their values ($c = 1$, and $\zeta^* = -0.03$) are close enough to the more recent determinations to ensure that the F values for unstable conditions are not in need of revision. For positive values of ζ, that is, for stable conditions, equation 12 can readily be integrated by noticing that, for $\alpha = 6$, $1 \gg \phi >$

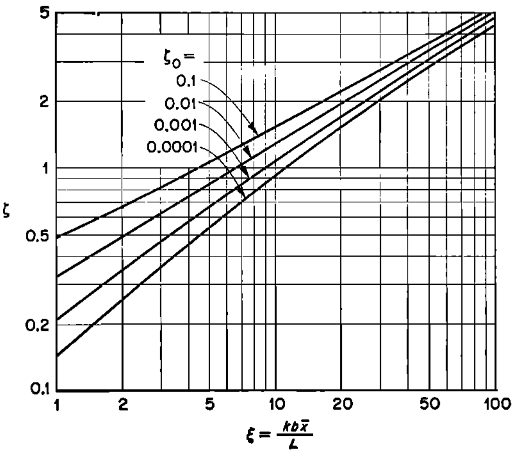

Fig. 1. Plot of $F(\zeta, \zeta_0) = \xi$, as defined by equation 16, for stable conditions (positive values of ζ).

0.955. This suggests that the following approximation be used for ϕ, wherever α is greater than about 5:

$$\bar{\phi} = \frac{1}{2}\left\{\left[\frac{1 + (\alpha - 1)\zeta}{1 + \alpha\zeta}\right]^{1/4} + 1\right\} \tag{15}$$

With this approximation, it is found directly that under stable conditions

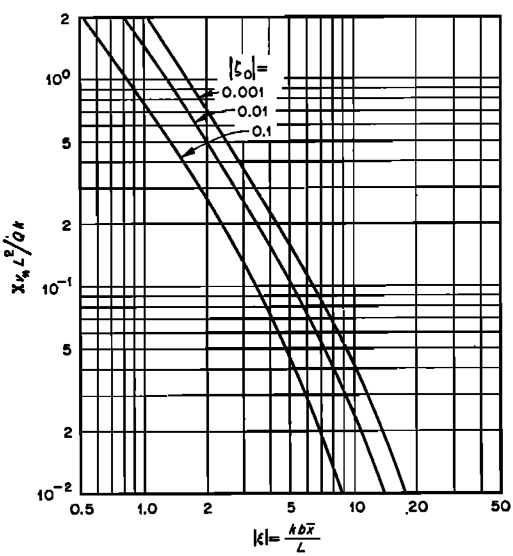

Fig. 2. Dimensionless axial concentration values Xv_*L^2/Qk plotted as a function of dimensionless downwind distance $|\xi| = kb\bar{x}/L$ for the case of unstable conditions.

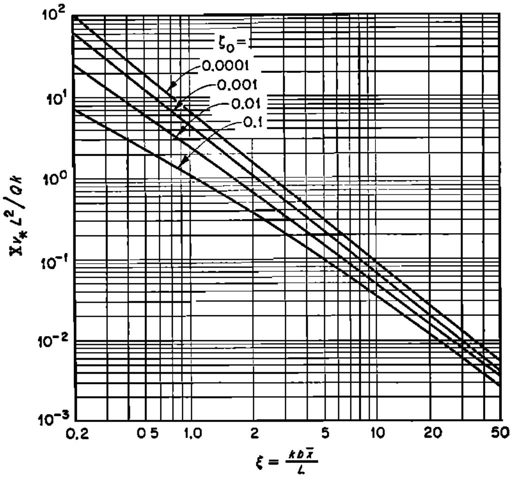

Fig. 3. Dimensionless axial concentration values Xv_*L^2/Qk plotted as a function of dimensionless downwind distance $\xi = kb\bar{x}/L$, for the case of stable conditions.

$$F(\zeta, \zeta_0) \simeq \frac{1}{\bar{\phi}}\left[\zeta \ln \frac{\zeta}{\zeta_0} - (\zeta - \zeta_0) + \frac{\alpha}{2}(\zeta - \zeta_0)^2\right] \tag{16}$$

Using this approximation, the function F has been calculated for a selection of ζ_0; the result, which is shown in Figure 1, differs somewhat from Kazansky and Monin's, owing to the greater weight assigned to the linear term in equation 8.

The various parameters, functions, and constants involved having been evaluated, it becomes possible to calculate values of concentration from a continuous point source located at the coordinate origin as a function of downwind distance $\bar{x}$. Figures 2 and 3 illustrate the results of the calculation, plotted in terms of the dimensionless parameters $\xi = kb\bar{x}/L$ and Xv_*L^2/Qk. Since z_0 is a constant for any particular set of experiments, ζ_0 must be a measure of atmospheric stability in the sense inverse to that of the stability length L; that is, ζ_0 equals zero for adiabatic conditions and is increasingly positive (or negative) for increasingly stable (or unstable) conditions.

It should perhaps be emphasized that the procedure described above, for arriving at these concentration curves, is valid independently of the exact nature of the universal functions or values of the constants involved. Should future surface-layer studies indicate that an adjustment in any of these is desirable, and at the moment this is a very active research area, the concentration curves will likewise need revision; but the method will still be valid.

Extension of these results to the case of a continuous infinite cross-wind line source follows, as in the adiabatic case, by integrating equation 5 (formally) with respect to y.

3. *Comparison with Prairie Grass data.* The very comprehensive program of dispersion field trials known as Project Prairie Grass is well suited to comparison with the above predictions. Observed ground level concentration values to a distance of 800 meters from a low-level, continuous point source were given by *Barad* [1958] and *Haugen* [1959], together with auxiliary data on the wind and temperature fields. *Cramer* [1957] has summarized this extensive body of data, and in particular has determined the power index m in the empirical formula

$$X \propto \bar{x}^m \tag{17}$$

best describing the observed downwind ground concentrations over various ranges of atmospheric stability and $\bar{x}$. Cramer chose, as a meteorological variable containing information on atmospheric stability, the standard deviation of the wind direction σ_A measured at a height of 2 meters. The close relationship between σ_A and turbulent dispersion in the lower layers has also been emphasized by *Pasquill* [1961]. The stability length L can be approximated as a function of σ_A by means of relationships given by *Inoue* [1959] in his equations 21 and 24. Since the roughness length z_0 for the Prairie Grass site was about 0.01 meter it becomes possible to identify Cramer's various stability categories approximately with values of $\zeta_0 = z_0/L$. Corresponding computed and observed power index values appear in Table 1.

The observed power index values are those by means of which Cramer has characterized the Prairie Grass data on downwind ground level concentrations. These data have been incorporated into a proposed empirical method for forecasting dispersion, discussed by *Pasquill* [1961] and *Meade* [1960]; see also *Gifford* [1961]. The calculated power index values were estimated directly from the curves in Figures 2 and 3.

TABLE 1. Comparison of Observed Point Source Concentration Power Law Index m with Theoretically Derived Values

Stability	ζ_0	$\bar{x}$, meters	m, observed	m, calculated
Very stable	+0.1	100–200	−1.3	−1.3
		400–800	−1.0	−1.3
Moderately stable	+0.01	50–100	−1.6	−1.6
		400–800	−1.4	−1.6
Near neutral	±0.001	50–100	−1.8	−1.6
		400–800	−1.8	−2.0
Moderately unstable	−0.01	50–100	−2.0	−2.2
		400–800	−2.6	−2.8
Very unstable	−0.1	50–100	−2.5	−2.8
		400–800	−3.3	−3.0

4. *Discussion.* The calculated index values agree with the observed values remarkably well. Under unstable conditions both the magnitude and the marked decrease of m that occurs with increasing $\bar{x}$ are found. Likewise under stable conditions the computed and observed values of m are close. In fact, all the significant observed features of the Prairie Grass data are well reproduced by the present theoretical model, with one exception. This is the slight increase (for example, from −1.3 to −1.0) observed in m with increasing $\bar{x}$ under very stable conditions. It is interesting in this connection that *Panofsky, Blackadar, and McVehil* [1960] have suggested that v_* and L may not uniquely define the wind profile under very stable conditions, because of the tendency for the lower atmosphere to become decoupled from the underlying ground surface then. Since it is reasonable to assume that this decoupling effect would tend to decrease the degree of turbulence in the lower layers, it would be expected that concentrations would in fact be higher than predicted. On the basis of this good agreement between the theoretical and observed axial concentration values, the hypothesis of Lagrangian similarity in the diabatic surface layer seems to be strongly supported. Arguing in the other direction, it can also be said that this agreement provides, at least over short distances of travel, good theoretical support for the empirical methods of estimating atmospheric dispersion to which reference has already been made.

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